

Existence and Stability of Pure Hopfions

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Joint work with

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The Faddeev-Hopf model

- Scalar field $\phi : \mathbb{R}^3 \rightarrow S^2 \subset \mathbb{R}^3$, subject to b.c. $\phi(\infty) = (0, 0, 1)$ (say)

$$\mathcal{E} = \int_{\mathbb{R}^3} \sum_j \frac{1}{2} |\partial_j \phi|^2 + \sum_{j,k} \frac{\alpha}{4} [\phi \cdot (\partial_j \phi \times \partial_k \phi)]^2$$

- Minimizers of \mathcal{E} called “hopfions”, labelled topologically by Hopf degree $\pi_3(S^2)$
- Knot-like (in some respects)

Geometrization

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Geometrization

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- Kähler form on $N = S^2$: $\omega(X, Y) = \langle JX, Y \rangle$
- Natural setting: $\phi : M \rightarrow N$
 - (M, g) Riemannian
 - (N, h, J) Kähler (meaning $d\omega = 0$ – automatic for N^2)

Two extreme limits

$$\mathcal{E} = \frac{1}{2} \int_M |d\phi|^2 + \alpha |\phi^* \omega|^2 = E_2 + \alpha E_4$$

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Heavily studied, rich theory

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- Programme: study variational problem for E_4 , compare with harmonic map theory

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$$E_4(\phi) = \frac{1}{2} \int_M |\phi^* \omega|^2 = \frac{1}{2} \|\phi^* \omega\|_{L^2}^2$$

- No finite energy non-vacuum solns if $M = \mathbb{R}^m$, $m \neq 4$ (Derrick's argument)

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- Critical maps $S^4 \rightarrow S^2$: **instantons** of FH model on \mathbb{R}^{3+1}
(in strong coupling limit)
- Open question: do they exist, for *any* N ?

First variation formula: the field equation

- Demand $E_4(\phi)$ is stationary under all smooth variations ϕ_t of ϕ

- Variation section $X = \left. \frac{\partial \phi_t}{\partial t} \right|_{t=0} \in \Gamma(\phi^{-1}TN)$

$$\left. \frac{d}{dt} \right|_{t=0} E_4(\phi_t) = \frac{1}{2} \left. \frac{d}{dt} \right|_{t=0} \langle \phi_t^* \omega, \phi_t^* \omega \rangle_{L^2}$$

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 \end{aligned}$$

- Holds for all sections X . Hence

$$d\phi \# \delta(\phi^* \omega) = 0$$

First variation formula: the field equation

- Now we know what $\boxed{\text{????}}$ is

$$\phi : M \rightarrow N \quad \text{critical point of } E_4 \iff \boxed{d\phi \# \delta(\phi^*\omega) = 0}$$

where

$$\begin{aligned} \delta &= (-1)^{m+mp+1} * d* = d^\dagger && \text{(coderivative)} \\ \langle \# \nu, Y \rangle &= \nu(Y) \quad \forall Y && \text{(musical isomorphism)} \end{aligned}$$

- Neat. But is it any use?

Immersive solutions ($\dim M \leq \dim N$)

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- But $H^2(S^4) = 0$ so $\phi^*\omega = 0$ (Hodge). Hence $E_4(\phi) = 0$

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- Generalizes: let $H^2(M) = 0$, $\dim M \leq \dim N$ and ϕ be a non-vacuum solution. Then the critical set of ϕ has nonzero measure in M .

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- Projection on a **warped** product: $\pi : P \times_f N \rightarrow N$ ($g = k + fh$)
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- Gives a smooth finite energy solution $(\mathbb{R}^4, g_{\text{fake}}) \rightarrow S^2$

$$g_{\text{fake}} = (1 + x_3^2 + x_4^2)^2(dx_1^2 + dx_2^2) + (1 + x_1^2 + x_2^2)^2(dx_3^2 + dx_4^2)$$

$$\pi : \mathbb{R}^4 \cong S_{\times}^2 \times S_{\times}^2 \rightarrow S^2$$

$[(\mathbb{R}^4, g_{\text{fake}})]$ is complete, Ricci positive, bounded scalar curvature

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- Interesting fact: for such maps

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- Such maps are solutions of the *full* FH model, $\mathcal{E} = E_2 + \alpha E_4$ for *all* α !

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- Examples

- Projection on a Kähler symmetric space, $\pi : G \rightarrow G/K$

e.g. $SU(k) \rightarrow \mathbb{C}P^{k-1} = SU(k)/S(U(1) \times U(k-1))$

- Hopf fibration, $\mathbb{C}^{k+1} \supset S^{2k+1} \rightarrow \mathbb{C}P^k$, $(z_0, z_1, \dots, z_k) \mapsto [z_0, z_1, \dots, z_k]$

Topological energy bound, $M^2 \rightarrow N$

- $\phi^*\omega = f_\phi * 1 \Rightarrow E = \frac{1}{2} \|f_\phi\|_{L^2}^2$

- Cauchy-Schwarz

$$\int_M \phi^*\omega = \int_M f_\phi * 1 \leq \|f_\phi\|_{L^2} (\text{Vol}(M))^{\frac{1}{2}}$$
$$\Rightarrow E \geq = \frac{1}{2\text{Vol}(M)} \left(\int_M \phi^*\omega \right)^2$$

- Attained iff $f_\phi = \text{const} \times 1$

- $\phi : M^2 \rightarrow S^2$ attains bound $\Rightarrow \phi^*\omega = \text{const} * 1$

$\Rightarrow \phi$ has no critical points $\Rightarrow \phi$ a local diffeomorphism

$\Rightarrow \phi$ a covering map

But $\pi_1(S^2) = 0$ so no nontrivial covering spaces

Hence $M^2 = S^2$ and ϕ a diffeomorphism

Hence $\deg \phi = \pm 1$

So $\phi^*\omega = \pm * 1 = \pm \omega$, $\Rightarrow \phi$ a symplectomorphism (up to reflexion)

Topological energy bound, $M^2 \rightarrow N$

- $T^2 \rightarrow T^2$

Bound attained in every class (linear periodic maps $\mathbb{C} \rightarrow \mathbb{C}$)

Automatically stable

- $\text{Id} : M^2 \rightarrow M^2$ attains bound

\Rightarrow Global minimizer in its class

Topological energy bound, $M^4 \rightarrow N$

$$E = \frac{1}{2} \int_M (*\phi^*\omega) \wedge \phi^*\omega \geq \frac{1}{2} \int_M \phi^*\omega \wedge \phi^*\omega = \frac{1}{2} \int_M \phi^*(\omega \wedge \omega)$$

- $\dim N = 2$
Trivial since $\omega \wedge \omega \equiv 0$
- $H^2(M) = 0$
 $\phi^*\omega$ closed, hence exact
 $\Rightarrow \phi^*\omega \wedge \phi^*\omega = (d\alpha) \wedge (d\alpha) = d(\alpha \wedge d\alpha)$
 \Rightarrow bound trivial
- Need $\dim N \geq 4$ and $H^2(M) \neq 0$: nontrivial but seldom attainable

Topological energy bound, $M^4 \rightarrow N$

- $\mathbb{C}P^2 \rightarrow \mathbb{C}P^2$

Attained iff $*\phi^*\omega = \pm\phi^*\omega$

$\phi^*\omega$ closed, self-dual \Rightarrow co-closed \Rightarrow harmonic $\Rightarrow \phi^*\omega = c\omega$

But then ϕ has no critical points \Rightarrow covering maps \Rightarrow diffeo

Hence ϕ symplectomorphism (up to reflexion)

- $\text{Id} : M^4 \rightarrow M^4$ attains bound
 \Rightarrow Global minimizer in its class

Stability

- In harmonic map theory, stability is characterized by the **Hessian**:

$$\text{Hess}_\phi(X, Y) = \frac{\partial^2 E[\phi_{s,t}]}{\partial s \partial t} \Big|_{s=t=0}, \quad X = \partial_s \phi_{s,t} \Big|_{s=t=0}, \quad Y = \partial_t \phi_{s,t} \Big|_{s=t=0}$$

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- ϕ is “stable” if $\text{Hess}_\phi(Y, Y) \geq 0$ for all Y
- Hessian **operator**: $\text{Hess}_\phi(X, Y) = \langle X, \mathcal{L}_\phi Y \rangle_{L^2} = \int_M h(X, \mathcal{L}_\phi Y) * 1$

Stability

- In harmonic map theory, stability is characterized by the **Hessian**:

$$\text{Hess}_\phi(X, Y) = \left. \frac{\partial^2 E[\phi_{s,t}]}{\partial s \partial t} \right|_{s=t=0}, \quad X = \left. \partial_s \phi_{s,t} \right|_{s=t=0}, \quad Y = \left. \partial_t \phi_{s,t} \right|_{s=t=0}$$

- ϕ is “stable” if $\text{Hess}_\phi(Y, Y) \geq 0$ for all Y
- Hessian **operator**: $\text{Hess}_\phi(X, Y) = \langle X, \mathcal{L}_\phi Y \rangle_{L^2} = \int_M h(X, \mathcal{L}_\phi Y) * 1$
- For $E = E_2$ (harmonic maps) \mathcal{L}_ϕ is the *Jacobi* operator:

$$\mathcal{L}_\phi Y = \bar{\Delta} Y - \sum_i R^N(Y, d\phi e_i) d\phi e_i$$

Stability properties depend strongly on curvature of N . Even $\text{Id} : M \rightarrow M$ can be unstable.

- What about $E = E_4$?

Stability

- **Thm** For $E_4(\phi)$, Hessian operator is

$$\mathcal{L}_\phi Y = -J[\nabla_{Z(\phi)}^\phi Y + d\phi \# \delta d(\phi^* \iota_Y \omega)]$$

where $Z(\phi) := \# \delta(\phi^* \omega) \in \ker d\phi$

$$(\phi^* \iota_Y \omega)(A) := \omega(Y, d\phi A) \quad \text{one-form on } M$$

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- **Cor** For any critical point ϕ of E_4 ,

$$\text{Hess}_\phi(Y, Y) = \int_M \omega(Y, \nabla_{Z(\phi)}^\phi Y) * 1 + \|d(\phi^* \iota_Y \omega)\|_{L^2}^2$$

In particular, if $Z(\phi) = 0$, then ϕ is stable.

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- **E.g.** All critical *immersions* are stable!

$\pi : P \times_f N \rightarrow N$ is stable

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 - Hopf : $S^3 \rightarrow S^2$ is stable
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 - $\text{spec } \mathcal{L}_{\text{Id}} = \text{spec } \Delta_1|_{\delta(\Lambda^3(N))}$ (= $\text{spec } \Delta_2$ for N^2, N^4)

Conclusion / prospects

- Energy functional $E_4(\phi) = \frac{1}{2} \|\phi^* \omega\|_{L^2}^2$ has beautiful variational equations:

$$0 = d\phi \# \delta(\phi^* \omega)$$
$$\text{Hess}_\phi(Y, Y) = \langle JY, \nabla_{Z(\phi)}^\phi Y \rangle_{L^2} + \|d(\phi^* \iota_Y \omega)\|_{L^2}^2$$

- Can give genuine insight into full FH model (esp. stability).
- Non-null-homotopic solution $S^4 \rightarrow S^2$, if it exists, would be a FH instanton
- Elusive, even for $S^4 \rightarrow N$.
- Are there any **unstable** solutions?

Conclusion / prospects

- A variation on the theme:

$$\mathcal{E}_{\text{Skyrme}} = E_2 + \alpha \hat{E}_4 = \int_M \frac{1}{2} |\mathrm{d}\phi|^2 + \frac{\alpha}{2} ((\mathrm{tr} \mathcal{D}_\phi)^2 - \mathrm{tr} \mathcal{D}_\phi^2) \quad [\text{Manton}]$$

where $\mathcal{D}_\phi : TM \rightarrow TM$, $g(X, \mathcal{D}_\phi Y) = h(\mathrm{d}\phi(X), \mathrm{d}\phi(Y))$.

- Makes sense for any (M, g) , (N, h)
- $M = \mathbb{R}^3$, $N = S^3 \Rightarrow$ (usual) Skyrme model
 $M = \mathbb{R}^3$, $N = S^2 \Rightarrow$ Faddeev-Hopf model
- $\alpha \rightarrow \infty$: pure Skyrme model, conformally invariant on M^4
- $M = S^4$, $N = S^3$: supports “instanton” solution in generator of $\pi_4(S^3)$