Manton's Geometric Vortex Dynamics

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Static Ginzburg-Landau theory

- Complex scalar field ϕ, gauge potential $A = A_1 dx_1 + A_2 dx_2$, $D\phi = d\phi - iA\phi$, $dA = B dx_1 \wedge dx_2$

$$V = \int \left(\frac{1}{2} B^2 + \frac{1}{2} |D\phi|^2 + \frac{\lambda}{8} (1 - |\phi|^2)^2 \right) dx_1 dx_2$$

- Finite V configurations may have non-zero winding at infinity $\Rightarrow$ magnetic flux quantization

$$\phi \sim e^{i\chi(\theta)}, \qquad D\phi \sim 0 \qquad \Rightarrow \qquad A \sim -i \frac{d\phi}{\phi} \sim \chi'(\theta) d\theta$$

$$\Rightarrow \int_{\mathbb{R}^2} dA = \int_{S^1_\infty} A = \chi(2\pi) - \chi(0) = 2n\pi$$

- Unit vortex solutions ($n = 1$)

$$\begin{aligned} \phi &= \sigma(r) e^{i\theta} = \left(1 + \frac{q(\lambda)}{2\pi} K_0(\sqrt{\lambda} r) + \cdots \right) e^{i\theta} \\ A &= a(r) d\theta = \left(1 + \frac{m(\lambda)}{2\pi} r K_1(r) + \cdots \right) \end{aligned}$$

Bogomol'nyi bound ($\lambda = 1$)

$$
\begin{aligned}
V &= \frac{1}{2}\int \left(B^2 + \overline{D_i\phi} D_i\phi + \frac{1}{2}(1-|\phi|^2)^2 \right) dx_1 dx_2 \\
&= \frac{1}{2}\int \left\{ (B - \frac{1}{2}(1-|\phi|^2))^2 + |D_1\phi + iD_2\phi|^2 + B - i(\partial_1(\overline{\phi}D_2\phi) - \partial_2(\overline{\phi}D_1\phi)) \right\} dx_1 dx_2 \\
&= \frac{1}{2}\int \left\{ (B - \frac{1}{2}(1-|\phi|^2))^2 + |D_1\phi + iD_2\phi|^2 \right\} dx_1 dx_2 + n\pi \\
\Rightarrow V &\geq n\pi
\end{aligned}
$$

- Attained if and only if

$$
\left.\begin{array}{l}
(D_1 + iD_2)\phi = 0 \\
B = \frac{1}{2}(1-|\phi|^2)
\end{array}\right\} \text{ Bogomol'nyi equations}
$$

- First order system of nonlinear PDEs

Taubes's reformulation

- Let $z = x_1 + ix_2$. Seek solution of Bog eqns with $\phi = 0$ at prescribed points $z_1, z_2, \ldots, z_n \in \mathbb{C}$ (vortex positions)
- Let $h = \log|\phi|^2$. Then h satisfies

$$\nabla^2 h + 1 - e^h = 4\pi \sum_{r=1}^{n} \delta(z - z_r)$$

- Taubes's Theorem: for each unordered list $[z_1, z_2, \ldots, z_n]$ of complex numbers there is a unique solution of this equation.
- Also gave estimates of asymptotic behaviour of h.
- Moduli space of static n-vortex solutions, $\mathsf{M}_n = \mathbb{C}^n/S_n = \mathbb{C}^n$

$$p(z) = (z - z_1)(z - z_2)\cdots(z - z_n) = z^n + q_1 z^{n-1} + q_2 z^{n-2} + \cdots + q_n$$

$q_1, \ldots, q_n$ good global coordinates.

Relativistic vortex dynamics

- Extend V to Minkowski space in Lorentz-invariant manner. $A = A_0 dx_0 + A_1 dx_1 + A_2 dx_2$,

$$S = \int_{\mathbb{R}^{2+1}} \left(\frac{1}{2} dA \wedge *dA + \frac{1}{2} D\phi \wedge *\overline{D\phi} - \frac{\lambda}{8} * (1 - |\phi|^2) \right)$$

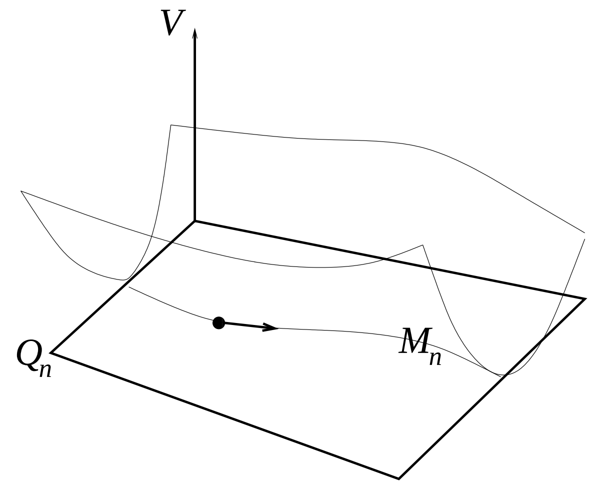
- Choose inertial frame, temporal gauge: $A_0 = 0$

$$\begin{aligned} S &= \int (T - V) dx_0 \\ \text{where} \quad V &= \int \left(\frac{1}{2} B^2 + \frac{1}{2} |D\phi|^2 + \frac{\lambda}{8} (1 - |\phi|^2)^2 \right) dx_1 dx_2 \\ T &= \frac{1}{2} \int \left(\dot{A}_1^2 + \dot{A}_2^2 + |\dot{\phi}|^2 \right) dx_1 dx_2 \end{aligned}$$

$B = \partial_1 A_2 - \partial_2 A_1$, $E_1 = \dot{A}_1$, $E_2 = \dot{A}_2$.

Relativistic vortex dynamics

- Initial value problem $(\phi(0), A(0)) \in \mathsf{M}_n$, $(\dot\phi(0), \dot A(0)) \in T_{(A(0),\phi(0))}\mathsf{M}_n$ small



- "Adiabatic" approximation: assume $(\phi(t), A(t)) \in \mathsf{M}_n$ for all t. Induced action

$$S| = \int (T| - V|)dx_0 = \int (\sum_{r,s} \gamma_{rs}(q)\dot q_r \dot{\overline{q}}_s - n\pi)dx_0$$

Geodesic motion on M_n w.r.t. metric

$$\gamma = \sum_{r,s} \gamma_{rs}(q) dq_r d\overline{q}_s, \qquad \gamma_{rs} = \int \left(\frac{\partial \phi}{\partial q_r}\overline{\frac{\partial \phi}{\partial q_s}} + \frac{\partial A_i}{\partial q_r}\frac{\overline{A_i}}{\partial q_s} \right) dx_1 dx_2$$

Geodesic approximation

- Vortex dynamics approximated by coupled ODE system: geodesic motion on (M_n, γ).
  Main task: understand γ.

- Samols (1992): γ is determined by behaviour of $|\phi|$ close to its zeros.
  Taubes: close to a simple zero $z = z_r$,

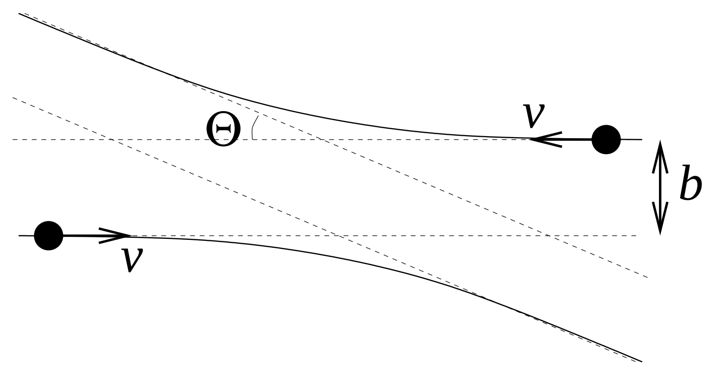
$$\log|\phi(z)|^2 = \log|z - z_r|^2 + a_r + \frac{1}{2}\overline{b}_r(z - z_r) + \frac{1}{2}b_r(\overline{z} - \overline{z}_r) + \cdots$$

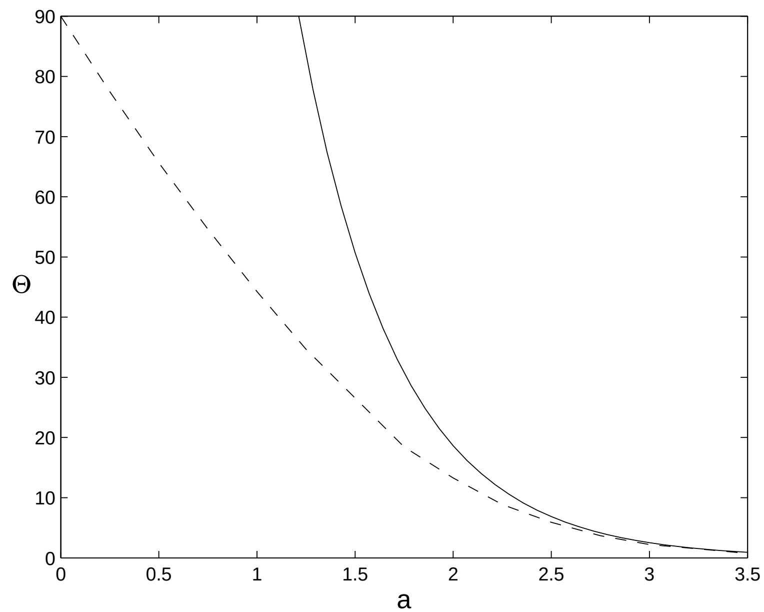
  Complex coefficients b_r depend in some (unknown) way on vortex positions $z_1, z_2, \ldots, z_n$. Samols's formula:

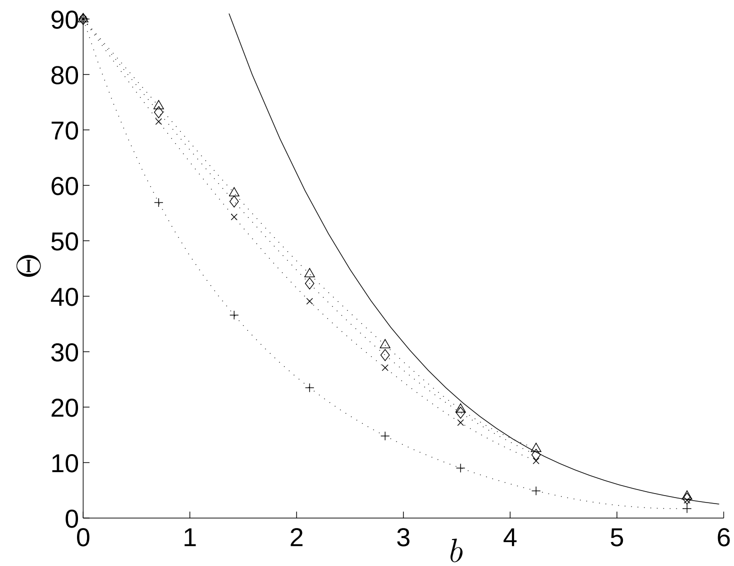
$$\gamma = \pi \sum_{r,s} \left(\delta_{r,s} + 2\frac{\partial b_r}{\partial z_s} \right) dz_r d\overline{z}_s$$

- Samols computed two-vortex metric numerically, studied scattering of two vortices

Two-vortex scattering







Head-on scattering

- $\mathsf{M}_2 = \mathbb{C}^2$

$$p(z) = (z - z_1)(z - z_2) = z^2 + q_1 z + q_2$$

- $\mathsf{M}_2^0 = \mathbb{C}$

$$p(z) = z^2 + q_2$$

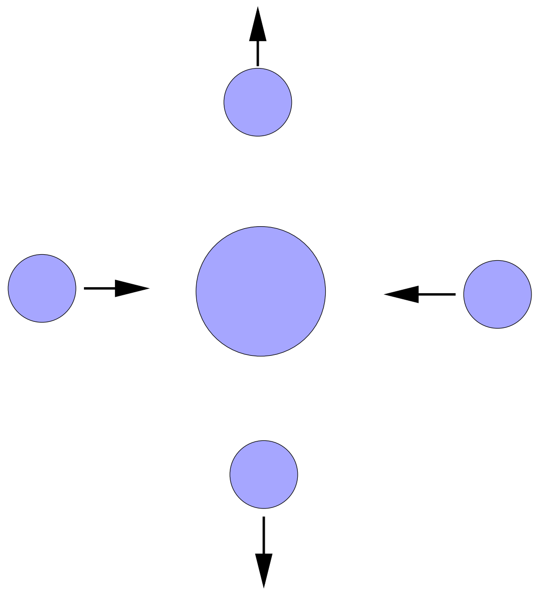
- Rotation symmetry:

$$(z_1, z_2) \longrightarrow (e^{i\alpha} z_1, e^{i\alpha} z_2) \quad \Rightarrow \quad q_1 = e^{2i\alpha} q_1$$

Hence

$$\gamma = F(|q_2|) dq_2 d\overline{q}_2$$

- Every radial line in q_2 plane is a geodesic, e.g. $q_2 = t$, $-\infty < t < \infty$



Asymptotic geometry

- Manton, Speight (2003): for $|z_r - z_s|$ large, all $r \neq s$

$$\gamma = \pi \sum_r dz_r d\overline{z}_r - \frac{q^2}{4\pi} \sum_{r \neq s} K_0(|z_r - z_s|)(dz_r - dz_s)(d\overline{z}_r - d\overline{z}_s) + \cdots$$

  where $q = q(1) = m(1)$ is a constant related to decay properties of a single vortex:

$$|\phi(r)| = 1 + \frac{q}{2\pi} K_0(r) + \cdots$$

  Tong conjectured (2002) $\frac{q}{2\pi} = -8^{\frac{1}{4}}$ (string duality)

- Physical interpretation: total energy of n moving point particles of mass π carrying scalar monopole charge q and magnetic dipole moment $m = q$, interacting via Klein-Gordon and Proca fields of mass 1.

- Forces between static "point vortices" exactly cancel. But attraction mediated by scalar field, repulsion by vector field $\Rightarrow$ moving vortices do exert forces on each other

First order vortex dynamics

- Kinetic energy linear in time derivatives

$$T = \gamma \int \left(\frac{i}{2}(\overline{\phi} D_0 \phi - \phi \overline{D_0 \phi}) + BA_0 + E_2 A_1 - E_1 A_2 - A_0 \right) dx_1 dx_2$$

  Action $S = \int (T - V) dx_0$ is gauge invariant (though Lagrangian density isn't)

- Field equations

$$\begin{aligned} (1) \qquad i\gamma D_0 \phi &= -\frac{1}{2} D_i D_i \phi - \frac{\lambda}{4}(1 - |\phi|^2)\phi \\ (2) \qquad \epsilon_{ij} \partial_j B &= -\frac{i}{2}(\overline{\phi} D_i \phi - \phi \overline{D_i \phi}) + 2\gamma \epsilon_{ij} E_j \\ (3) \qquad 2B &= 1 - |\phi|^2 \end{aligned}$$

  First order flow (initial data $(\phi(0), A(0))$). Conserves energy V (*not* $T + V$).

- Static solutions? Can't just solve static GL equation, as vortex solutions don't satisfy constraint (3)
  Critical coupling, $\lambda = 1$: vortices *do* satisfy (3) — have space M_n of static n-vortex solutions.

Moduli space approximation

- $\lambda \approx 1$: Given $(\phi(0), A(0)) \in \mathsf{M}_n$, what is subsequent motion?
- $(\phi(t), A(t))$ can't move far from M_n (conservation of V) and must satisfy (3) for all t.
- Suggests adiabatic approx: $(\phi(t), A(t)) \in \mathsf{M}_n$ for all t, position evolves slowly according to induced action

$$\begin{aligned} S|_{\mathsf{M}_n} &= \int (T|_{\mathsf{M}_n} - V|_{\mathsf{M}_n}) dx_0 = \int L_{red}\, dx_0 \\ L_{red} &= \sum_a \mathcal{A}(q)_a \dot{q}_a - V(q) \end{aligned}$$

$q_1, \ldots, q_{2n}$ an arbitrary coord system on M_n

- n-vortex dynamics again approximated by system of nonlinear ODES:

$$\sum_a \mathcal{B}_{ab} \dot{q}_a = -\frac{\partial V}{\partial q_b} \qquad (*), \qquad\qquad \mathcal{B}_{ab} = \frac{\partial \mathcal{A}_b}{\partial q_a} - \frac{\partial \mathcal{A}_a}{\partial q_b}$$

Moduli space approximation

- n-vortex dynamics again approximated by system of nonlinear ODES:

$$\sum_a \mathcal{B}_{ab}\dot{q}_a = -\frac{\partial V}{\partial q_b} \qquad (*), \qquad\qquad \mathcal{B}_{ab} = \frac{\partial \mathcal{A}_b}{\partial q_a} - \frac{\partial \mathcal{A}_a}{\partial q_b}$$

- More geometrically: one-form $\mathcal{A} = \sum_a \mathcal{A}_a dq_a$, $\mathcal{B} = d\mathcal{A}$ is a symplectic form on M_n, and $(*)$ is Hamiltonian flow on $(\mathsf{M}_n, \mathcal{B})$ with Hamiltonian V.
- Given $F : \mathsf{M}_n \to \mathbb{R}$, define symplectic gradient X_F such that $\mathcal{B}(\cdot, \mathsf{X}_F) = dF(\cdot)$. Then $(*)$ is

$$\dot{q} = \mathsf{X}_V.$$

Moduli space approximation

- There's another natural symplectic form on M_n, the Kähler form of the L^2 metric γ.
- Recall M_n is a complex manifold: have a natural linear map

$$J : T_q\mathsf{M}_n \to T_q\mathsf{M}_n \qquad \text{s.t.} \qquad J^2 = -1$$

  "multiply vector by i"

- Fact: J is an isometry $\qquad \gamma(JX, JY) = \gamma(X, Y) \qquad$ (metric is Hermitian).
- Hence, can define Kähler form $\qquad \omega(X, Y) = \gamma(JX, Y)$
  [Check: $\omega(Y, X) = \gamma(JY, X) = \gamma(JJY, JX) = -\gamma(Y, JX) = -\omega(X, Y)$.]

- Fact: $d\omega = 0$ (metric is Kähler). Follows from Samols's formula for γ.
- Nondegenerate by nondegeneracy of γ. Hence a symplectic form.

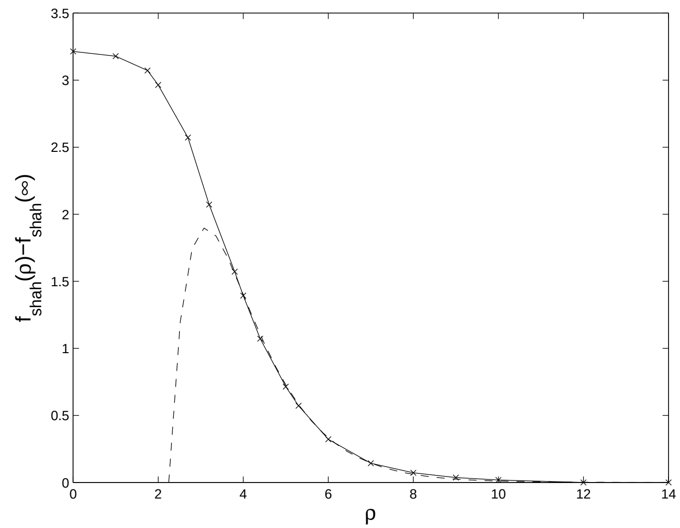
Moduli space approximation

- Question: is there a relationship between $\mathcal{B}$ and ω?
- Romao, Speight (2004): $\mathcal{B} = -2\gamma\omega$
  Hence reduced dynamics is Hamiltonian flow in (M_n, ω) with Hamiltonian $-V/2\gamma$.
- "Know" what $\mathcal{B}$ is. What about $V|_{\mathsf{M}_n}$?

$$
\begin{aligned}
V &= \int \left(\frac{1}{2}B^2 + \frac{1}{2}|D\phi|^2 + \frac{1}{8}(1-|\phi|^2)^2\right) dx_1 dx_2 + \frac{\lambda-1}{8}\int (1-|\phi|^2)^2 dx_1 dx_2 \\
&= n\pi + \frac{\lambda-1}{8}\int (1-|\phi|^2)^2 dx_1 dx_2 \\
&= n\pi + \frac{\lambda-1}{8} f_{Shah}(z_1, z_2, \ldots, z_n)
\end{aligned}
$$

Moduli space approximation

- Shah computed f_{Shah} numerically, for $n = 2$. Depends only on $\rho = |z_1 - z_2|$. Monotonically decreasing.



- $\lambda > 1$, $\gamma > 0$: vortex pairs orbit their COM anticlockwise at constant separation and angular velocity.

Well-separated vortices

- Assume $|z_r - z_s|$ large for all $r \neq s$. Then

$$\omega = \pi \sum_{r=1}^{n} dz_r \wedge d\bar{z}_r + \text{exponentially small}$$

- For $\lambda = 1+\varepsilon$, compare $V|_{\mathsf{M}_n} = n\pi + \frac{\lambda-1}{8}\int(1-|\phi|^2)^2 dx_1 dx_2$ with interaction energy of well separated point vortices

$$\begin{aligned} U_{int} &= \sum_{r<s} \mathcal{U}_{pv}(|z_r - z_s|) \\ \mathcal{U}_{pv}(\rho) &= -\frac{q(\lambda)^2}{2\pi} K_0(\sqrt{\lambda}\rho) - m(\lambda)^2 K_0(\rho) \end{aligned}$$

Expand in ε, equate leading terms. Suggests

$$\begin{aligned} V|_{\mathsf{M}_n} &\sim \text{const} + \sum_{r<s} \mathcal{U}(|z_r - z_s|) \\ \mathcal{U}(\rho) &= \frac{q(1)^2}{4\pi}[\rho K_1(\rho) - \nu K_0(\rho)] \end{aligned}$$

where $\nu = 4(q'(1) - m'(1))/q(1) \approx 2.7$

Well-separated vortices

- Flow along symplectic gradient of $-V|_{\mathsf{M}_n}/2\gamma$

$$\dot{z}_r = \sum_{s \neq r} F(|z_r - z_s|)(z_r - z_s), \qquad F(\rho) = -\frac{i}{2\pi\gamma}\frac{\mathcal{U}'(\rho)}{\rho}$$

  Similar to motion of fluid point vortices, geostrophic vortices

$$\mathcal{U}_{fluid}(\rho) = \log \rho, \qquad \mathcal{U}_{geo}(\rho) = K_0(\rho)$$

- 2-vortex dynamics trivial
- 3-vortex dynamics can be understood in great detail. Surprising fact: given any choice of initial vortex positions z_1, z_2, z_3 there is some time when the vortex triangle is isoceles

Well-separated vortices

- Rotating vortex polygons: Let $\xi = e^{2\pi i/n}$, consider IVP $z_r(0) = \sigma \xi^r$, $r = 1, 2, \ldots, n$, $\sigma > 0$ large: vortices at the vertices of a regular n-gon. Polygon rotates with constant angular velocity

$$\Omega = \sum_{r=1}^{n-1} F(|1 - \xi^r|\sigma)$$

- Filled vortex polygons: Let $\xi = e^{2\pi i/(n-1)}$, consider IVP $z_r(0) = \sigma \xi^r$, $r = 1, 2, \ldots, n-1$, $z_n(0) = 0$, $\sigma > 0$ large: vortices at the vertices of a regular $(n-1)$-gon, one vortex at centre. Polygon rotates with constant angular velocity

$$\Omega = F(\sigma) + \sum_{r=1}^{n-2} F(|1 - \xi^r|\sigma)(1 - \xi^r)$$

Well-separated vortices

- Spectral stability analysis:

Rotating n-gons

n	2	3	4	5	6	7	≥ 8
GL	1	1	1	1	0	0	0
fluid	1	1	1	1	1	1	0
geostrophic	1	1	1	1	0	0	0

Rotating filled $(n-1)$-gons

$n-1$	2	3	4	5	6	7	8	9	≥ 10
GL	0	1	1	1	1	1	0	0	0
fluid	0	1	1	1	1	1	1	1	0
geostrophic	0	0	1	1	1	1	0	0	0

- Krusch and Sutcliffe (2004): numerical simulation of full field equations. Good agreement with moduli space approx (Romao, Speight) for $0.9 \leq \lambda \leq 1.1$. Important radiative effects for λ far from 1.
- Symmetry of rotating polygons stable as predicted, except pentagon.