Nonhelical inverse transfer of a decaying turbulent magnetic field

Cosmological magnetic fields
Turbulent decay
Nonuniversality of MHD
Weak and strong turbulence
Helical, nonhelical, hydro

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Motivation

Early universe

Energy momentum tensor

\[ T^{\mu \nu} = (p + \rho) U^{\mu} U^{\nu} + pg^{\mu \nu} \]

\[ + \frac{1}{4 \pi} \left( F^{\mu \sigma} F_{\nu \sigma} - \frac{1}{4} g^{\mu \nu} F_{\lambda \sigma} F^{\lambda \sigma} \right), \]

Conformal time, rescaled equations

\[ \tilde{t} = \int dt / R. \quad \tilde{S} = (p + \rho) \gamma^2 v. \]

\[ \frac{\partial \tilde{S}}{\partial \tilde{t}} = - (\nabla \cdot v) \tilde{S} - (v \cdot \nabla) \tilde{S} - \nabla \tilde{p} + \tilde{J} \times \tilde{B}. \]

\[ \frac{\partial \tilde{B}}{\partial \tilde{t}} = - \nabla \times \tilde{E}, \quad \nabla \cdot \tilde{B} = 0, \]

the MHD equations in an expanding universe with zero curvature are the same as the relativistic MHD equations in a nonexpanding universe, provided the dynamical quantities are replaced by the scaled “tilde” variables, and provided conformal time \( \tilde{t} \) is used. The effect of this is, as usual, that...
Helical vs nonhelical 3-D decay

Initial slope
\( E \sim k^4 \)

Christensson et al.
(2001, PRE 64, 056405)
Helical decay law: Biskamp & Müller (1999)

\[ H = EL = \text{const} \]

\[ \varepsilon = \frac{U^3}{L} = \frac{E^{3/2}}{L} \]

\[ \varepsilon = -\frac{dE}{dt} \]

\[ \varepsilon = -\frac{dE}{dt} = \frac{E^{3/2}}{L} = \frac{E^{5/2}}{H} \]

\[ E \propto t^{-2/3} \]

\[ L \propto t^{+2/3} \]
Nonhelical Inverse Transfer of a Decaying Turbulent Magnetic Field

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\[ E_{WT}(k, t) = C_{WT}(\epsilon \nu_A k_M)^{1/2} k^{-2} \]
Weak MHD turbulence, because $B$ strong

Lee, Brachet, Pouquet, Mininni, Rosenberg (2010)
Inverse transfer similar to helical MHD

\[ T_{kpq} = \langle J^k \cdot (u^p \times B^q) \rangle = -\langle u^p \cdot (J^k \times B^q) \rangle \]

Nonhelical gain ½ of helical case

Gain from SS \( B \) Mediated by LS \( u \)

Kinetic gain From \( B \) field
On Inverse Cascades and Primordial Magnetic Fields

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Next we want to use the well known self-similarity property of the non-relativistic Navier-Stokes or MHD-equations,

\[ x \rightarrow lx, \quad t \rightarrow l^{1-h}t, \quad v \rightarrow l^h v, \quad \nu \rightarrow l^{1+h} \nu, \quad \mathbf{B} \rightarrow l^h \mathbf{B}, \quad \eta \rightarrow l^{1+h} \eta, \]  

(5)

where \( \nu \) is the kinetic and \( \eta \) is the Ohmic diffusion. Using the substitutions \( x = lx' \) and \( y = ly' \), we obtain from eqs. (2) and (3)

\[
E(k/l, l^{1-h}t, Ll, K/l) = l^d \frac{2\pi k^2}{(2\pi)^3} \int_{2\pi/K}^{L} d^3x' d^3y' \ e^{ik(x'-y')} <v(lx', l^{1-h}t) \ v(ly', l^{1-h}t) > \\
= l^{d+2h} E(k, t, L, K).
\]

(6)

primordial magnetic fields, and for the effect of diffusion. In general, if the initial spectrum is \( k^\alpha \), then in the “inertial” range, for \( \alpha > -3 \) there is an inverse cascade, whereas for \( \alpha < -3 \) there is a forward cascade.
**Does initial spectrum determine decay?**

\[ E_K(k, t) \sim E_M(k, t) \sim k^{\alpha \psi(k^{(3+\alpha)/2}t)}. \]  \hspace{1cm} (4)

Integrating over \( k \) yields the decay law of the energies as

\[ \mathcal{E}_K(t) = \int E_K(k, t) \, dk \sim \int k^{\alpha \psi(k^{(3+\alpha)/2}t)} \, dk. \]  \hspace{1cm} (5)

Introducing \( \kappa = kt^q \) with \( q = 2/(3 + \alpha) \), we have

\[ \mathcal{E}_K(t) \sim t^p \int k^{\alpha \psi(\kappa)} \, d\kappa, \]  \hspace{1cm} (6)

where \( p = (1 + \alpha)q \). The integral scales like \( k_K \sim t^q \) with \( q = 2/(3 + \alpha) \). Several parameter combinations are given in Table II.

\[
\begin{array}{ccc}
\alpha & p & q \\
4 & 10/7 & 2/7 \\
3 & 8/6 & 2/6 \\
2 & 6/5 & 2/5 \\
1 & 4/4 & 2/4 \\
0 & 2/3 & 2/3 \\
\end{array}
\]

\[ q = 1 - p/2 \]
Rescaled spectra: self-similar

Alternative interpretation of Olesen’s scaling relation

Christensson et al.
(2001, PRE 64, 056405)

Initial slope

$E \sim k^4$

FIG. 2. The magnetic scaling function $g_M(k\xi)$ described in the text, Eq. (13), versus $k\xi$. The straight lines indicate the power laws $\propto (k\xi)^{4.0}$ and $\propto (k\xi)^{-2.5}$, respectively.

$$E_M(k,t) = \xi(t)^{-q}g_M(k\xi). \quad (13)$$
Revised interpretation

\[ E(k, t) = \xi \phi(k \xi) \]

with integral scale \( \xi \)

\[ \xi = \xi(t) \propto t^q \]

and \( q \) determined by physics

\[ \beta = -3 + 2/q \]

from dimensional arguments

\[ \beta \quad p \quad q \quad \text{physics} \]

\begin{array}{llll}
4 & 10/7 & 2/7 & \int u^2 r^4 \, dr = \mathcal{L} = \text{const} \sim \ell^7 \tau^{-2} \quad \text{(Loitsiankii)} \\
3 & 8/6 & 2/6 & \\
2 & 6/5 & 2/5 & \int u^2 r \, dr = \mathcal{C} = \text{const} \sim \ell^5 \tau^{-2} \quad \text{(Saffman)} \\
1 & 4/4 & 2/4 & \langle A_{2D}^2 \rangle = \text{const} \sim \ell^4 \tau^{-2} \quad \text{(to be confirmed)} \\
0 & 2/3 & 2/3 & \langle A \cdot B \rangle = \text{const} \sim \ell^3 \tau^{-2} \quad \text{(Biskamp & Müller)}
\end{array}
Scaling relations

FIG. 1: Kinetic energy spectra in a hydrodynamic simulation (a), compared with magnetic (solid) and kinetic (dashed) energy spectra in a hydromagnetic simulation without helicity (b) and (c), and with (d). Panels (e)–(h) show the corresponding collapsed spectra obtained by using $\beta_M = 3$ (e), $\beta_M = 2$ (f), $\beta = 1$ (g), and $\beta = 0$ (h). In (f) we used $\beta_K = 1 \neq \beta_M$. 
Conclusions

• Helicity slows down decay
• Large scale energy increases
• Nonhelical inverse transfer
• Revised interpretation to Olesen
• Self-similar spectra
• Confirmed now by others (Berera & Linkmann 2015 Zrake 2015)