Salient features of fluid turbulence.

Experimental aspects:

- Structure function

\[ \delta u_1(x) = \frac{u(x + l) - u(x)}{l} \]

\[ S_1(l) = \left( \left[ \delta u_1(l) \right]^2 \right) \]

- Scaling of \( S_2(l) \)

\[ u(x + l) - u(x) \sim \# l \]

for \( l \ll l_1 \)

velocity differences are "smooth" at small scales.

Here "smooth" \( \equiv \) expandable in a Taylor series.

- Energy spectra.

\[ E(k) \sim \frac{1}{k^4}, k_0 \sim \frac{1}{L}, k_3 \sim \frac{1}{l} \]

- Multiscaling.

\[ S_p(l) \sim l^p \quad \text{with} \]

\[ S_p(S_2(l)) \sim l^p \quad \text{--- Karman - Howarth} \]

- Zeroth law of turbulence.

\[ \varepsilon = \frac{1}{2} \left( \frac{Du}{dt} \right) \]

\[ \varepsilon = - \frac{d}{dt} \frac{E}{E} \]
\[ \varepsilon = -2\nu \Omega = -2\nu \langle |\nabla \times \mathbf{u}(\mathbf{x})|^2 \rangle \]
\[ = -2\nu \int \frac{[\nabla \times \mathbf{u}(\mathbf{x})]^2}{\mathbf{v}} \, d^3x \]

As \( \nu \to 0 \), \( \varepsilon(\nu) \to \varepsilon > 0 \).

"Diss"

As \( \varepsilon \) becomes smaller and smaller the fluid velocity becomes less and less regular such that the product tends to a finite positive number.

"Dissipative Anomaly"

This is a basic hypothesis behind Kolmogorov's theory of turbulence and all subsequent phenomenological theories of turbulence.

We shall see how difficult aspects of this basic

"To see the world in a grain of sand
And heaven in a wild flower"

William Blake
Auguries of Innocence.

An important goal of statistical theories of turbulence is to understand these features of turbulence, particularly in multiscale.
Kraichnan model of passive scalar.

\[ \partial_t \theta + (\mathbf{u} \cdot \nabla) \theta = \kappa \nabla^2 \theta + \mathbf{f}. \]

The structure for

where the velocity \( \mathbf{u} \) is Gaussian and white-in-time, such that

\[ \langle \mathbf{u} \rangle = 0, \quad \langle \mathbf{u} \mathbf{u}^T \rangle = D \]

\[ \langle \nabla \mathbf{u} \rangle = \quad \langle \nabla \mathbf{u} (x,t) \nabla \mathbf{u} (x',t') \rangle = \delta (x-x') \delta (t-t') \]

\[ \langle \nabla \mathbf{u} \nabla \mathbf{u} (x,t) \rangle = \quad \mathcal{D}^{\Delta} (x) \delta (t-t') \]

or in Fourier space.

\[ \hat{\mathbf{u}} (k,t) = \mathcal{F} \int \mathbf{u} (x,t) e^{i \mathbf{k} \cdot \mathbf{x}} d^3x \]

\[ \mathbf{u} (x,t) = \frac{1}{2\pi} \int \hat{\mathbf{u}} (k,t) e^{-i \mathbf{k} \cdot \mathbf{x}} d^3k \]

and

\[ \langle \hat{\mathbf{u}} (k,t) \hat{\mathbf{u}} (k',t') \rangle = \mathcal{D}^{\Delta} (k \cdot k') \delta (t-t') \]

Translational invariance:

\[ D^{\Delta} (x, x') = D^{\Delta} (x-x') \]

\[ \Rightarrow \quad \hat{D}^{\Delta} (k, k') = \hat{D}^{\Delta} (k) \]

\[ \hat{D}^{\Delta} (k) = \int D^{\Delta} (x) e^{i \mathbf{k} \cdot \mathbf{x}} d^3x. \]
incompressibility.

\[
\hat{D}(k) = P^{(2)}(k) \frac{e^{-(\gamma k)^2}}{(k^2 + k_0^2)^{d+1/2}}
\]

with \(0 \leq \xi \leq 2\)

\[
k_0 \sim \frac{1}{L}, \quad \xi \sim \frac{1}{k_d}
\]

\[
\hat{D}^{(2)}(k) \sim - (d+3) k
\]

\[
P^{(2)}(k) = \delta^{(2)} - \frac{k^2}{k_d^2}
\]

This mimics the turbulent velocity field.

But not in the following way:

- while-in-time is a
- Gaussian nature of PDF of \(u\).

The main point in the study of the whole model is that although the statistics of \(u\) is Gaussian, we shall obtain a non-Gaussian, multiscaling statistics for the passive scalar field. The statistical properties of the scalar will mimic that of turbulent velocity field.

Karman-Howarth:

\[
\langle (\delta u_{11})^3 \rangle = -\frac{4}{5} \varepsilon \gamma
\]

\[
\langle \delta u_{11} (\delta \theta)^2 \rangle = -\frac{4}{3} \varepsilon^6 \gamma
\]

\[
\varepsilon = \nu \langle (\nabla u)^2 \rangle
\]

\[
\varepsilon^6 = \kappa \langle (\nabla \theta)^2 \rangle
\]
\[
\begin{align*}
\frac{\partial}{\partial t} \theta + \frac{\partial}{\partial x} \frac{\partial}{\partial t} \theta = \frac{\partial}{\partial t} \theta + \frac{\partial}{\partial x} \theta = \alpha \frac{\partial}{\partial t} \theta + f \frac{\partial}{\partial x} \theta
\end{align*}
\]

We begin by first studying the structure function of passive scalar.

\[
S_2(x,t) = \left< \left[ \theta(x+t,t) - \theta(x,t) \right]^2 \right>
\]

\[
= 2 \left< \theta(x,t) \right> - \left< \theta(x+t,t) \theta(x,t) \right>
\]

\[
= 2 \left[ \theta_2(x,t) - 2 \theta_2(x,t) \right]
\]

\[
c_2(x,t) = \left< \theta(x+t,t) \theta(x,t) \right>
\]

The

\[
\frac{\partial}{\partial t} c_2(x,t) = \left< \left[ \frac{\partial}{\partial t} \theta(x+t,t) \right] \theta(x,t) \right> + \left< \theta(x+t,t) \theta_2(x,t) \right>
\]

\[
= \left< \theta(x,t) \right> + \left< \theta_2(x,t) \right>
\]

\[
\left< \theta(x,t) \right> = \left< \theta(x,t) \right>
\]

\[
\left< \theta_2(x,t) \right> = \left< \left[ - \frac{\partial^2}{\partial x^2} \theta + \alpha \frac{\partial}{\partial t} \theta + f \frac{\partial}{\partial x} \theta \right] \theta \right>
\]

\[
= - \left< \left[ \frac{\partial}{\partial x} \theta \right] \theta \right> + \alpha \left< \theta \frac{\partial}{\partial t} \theta \right> + \left< \frac{\partial}{\partial x} \theta \right>
\]

\[
= - \frac{\partial}{\partial x} \left< \theta \theta \theta \right> + \alpha \left< \theta \theta \theta \right> + \left< \frac{\partial}{\partial x} \theta \right>
\]

\[
= - \frac{\partial}{\partial x} \left< \theta \theta \theta \right> + \alpha \left< \theta \theta \theta \right> + \left< \frac{\partial}{\partial x} \theta \right>
\]

\[
= - \frac{\partial}{\partial x} \left< \theta \theta \theta \right> + \alpha \left< \theta \theta \theta \right> + \left< \frac{\partial}{\partial x} \theta \right>
\]

\[
+ \frac{\partial}{\partial x} \left< \theta \theta \theta \right> + \left< \frac{\partial}{\partial x} \theta \right>
\]

\[
+ \alpha \left< \theta \theta \theta \right> + \left< \frac{\partial}{\partial x} \theta \right>
\]

\[
= - \frac{\partial}{\partial x} \left< \theta \theta \theta \right> + \alpha \left< \theta \theta \theta \right> + \left< \frac{\partial}{\partial x} \theta \right>
\]

\[
+ \frac{\partial}{\partial x} \left< \theta \theta \theta \right> + \left< \frac{\partial}{\partial x} \theta \right>
\]

\[
+ \alpha \left< \theta \theta \theta \right> + \left< \frac{\partial}{\partial x} \theta \right>
\]
Here \( \langle \rangle \) denotes averaging over both the velocity and the force. We already know the statistics of \( \theta, u \); we now specify the statistics of \( f \).

\( f \) is Gaussian, white-in-time, zero mean

\[ \langle f(x,t) \rangle = 0 \]

and

\[ \langle f(x,t) f(x',t') \rangle = \delta(x-x') \delta(t-t') \]

and

\[ \hat{f}(k) \equiv \hat{f}(k) = \int \hat{f}(\mathbf{r}) e^{i \mathbf{k} \cdot \mathbf{r}} d^3 \mathbf{r} \]

\[ \hat{f}(k) = \hat{f}(k) \]

- isotropic forcing.

consider

\[ \langle u' \theta' \theta \rangle = \langle u'_i \theta'[u,f] \theta[u,f] \rangle \]

\[ = \frac{\langle u' \cdot A[u,f] \rangle}{\langle u'_i u'_j \rangle \langle \delta A \delta u''_j \rangle} \]

Claim:

\[ \langle u'_i A[u,f] \rangle = \delta^{(3)}(x-x',t-t') \]

\[ = \langle u'_i u'_j \rangle \langle \delta A \delta u''_j \rangle \]
Proof.

Let $x$ be a Gaussian random variable. And $f(x)$ be some function of $x$. Then,

$$
\langle \sqrt{2\pi \sigma^2} \rangle = e^{-\frac{x^2}{2\sigma^2}} \quad \text{with} \quad \sigma^2 = \frac{1}{2}\int_{-\infty}^{\infty} x^2 f(x) p(x) \, dx
$$

Now for $x = (x_1, x_2, \ldots, x_N)$

Then

$$
\langle x_i f(x_j) \rangle = \langle x_i x_j \rangle \langle \frac{\partial f}{\partial x_j} \rangle
$$

Generalising to functions

$$
\langle u(x) f[u] \rangle = \langle u(x) u(y) \rangle \langle \frac{\delta f}{\delta u(y)} \rangle
$$

Then

$$
\langle u(x', t') A[u, f] \rangle = \langle u(x', t') u(x, t) \rangle \langle \frac{\delta A}{\delta u(x, t')} \rangle
$$
Then

\[
\langle u(x', t') \theta(x', t') \theta(x, t) \rangle
\]

\[
= \langle u(x', t) u(x'', t'') \theta(x', t') \theta(x, t) \rangle \frac{\delta}{\delta u(x'', t'')}
\]

\[
= \delta^\rho(\tilde{x}' - \tilde{x}'') \delta(t - t'') \langle \frac{\delta}{\delta u(x'', t'')} \theta(x', t') \theta(x, t) \rangle
\]

\[
= \delta^\rho(\tilde{x}' - \tilde{x}'') \langle \frac{\delta}{\delta u(x', t)} \theta(x', t) \theta(x, t) \rangle
\]

How to evaluate this functional derivative?

Let us look at a simple example:

\[
\frac{\delta}{\delta u} \phi = \frac{\partial}{\partial t} \phi = u \phi(x) + \phi \phi.
\]

\[
\phi(t) = \phi(t_0) + \int_{t_0}^{t} u(t'') \delta [\phi(t'')] dt'' + \phi \phi(t)
\]

\[
\frac{\delta \phi(t)}{\delta u(t')} = \int_{t_0}^{t} \frac{\delta u(t'')} \delta [\phi(t'')] dt''
\]

\[
= \delta [\phi(t')] \quad \text{for} \quad t \geq t'
\]

\[
= 0 \quad \text{for} \quad t < t'
\]
\[ A = \frac{\theta(x', t)}{\Delta u^\rho(x'', t)} \theta(x, t) \]

\[ = \theta(x, t) \frac{\delta \theta(x', t)}{\delta u^\rho(x'', t)} + \theta(x', t) \frac{\delta \theta(x, t)}{\delta u^\rho(x'', t)} \]

\[ = \nabla \theta(x, t) \]

\[ \partial_t \theta(x, t) = -u(x, t) \cdot \nabla \theta(x, t) + \ldots \]

\[ \theta(x, t) = -\int u(x', t') \cdot \nabla \theta(x', t') \, dt' \]

\[ \frac{\delta \theta(x, t)}{\delta u^\rho(x'', t)} = -\int u_\rho(x', t') \frac{\partial}{\partial x} \theta(x', t') \, dt' \]

\[ \frac{\delta \theta(x, t)}{\delta u^\rho(x'', t)} = -\delta(x - x'') \frac{\partial}{\partial x} \theta(x, t') \]

\[ A = -\theta(x, t) \frac{\partial}{\partial x} \theta(x', t) + \theta(x', t) \frac{\partial}{\partial x} \theta(x, t) \]

\[ \langle u_\rho(x', t) \theta(x', t) \theta(x, t) \rangle \]

\[ = \left[ - \frac{\partial}{\partial x} \theta(x) \theta(x') - \frac{\partial}{\partial \rho} \theta(x) \theta(x') \right] D^{\rho} (x - x') \]

\[ = -D^{\rho} (x - x') \left( \frac{\partial}{\partial \rho} + \frac{\partial}{\partial x} \right) \langle \theta(x) \theta(x') \rangle \]

\[ \langle \partial_t \theta \rangle = +D^{\rho} (x - x') \left[ \frac{\partial}{\partial x} \theta(x') \frac{x'}{\theta'} + \frac{\partial}{\partial \rho} \theta(x') \right] \langle \theta(x) \theta(x') \rangle \]
\[ a_{t} c_{2} (x' - x, t) = D^{\alpha} (x - x') \left[ \frac{\partial}{\partial x_{\alpha}} \frac{\partial}{\partial x'_{\beta}} + \frac{\partial}{\partial x'_{\alpha}} \frac{\partial}{\partial x_{\beta}} \right] c_{2} \]

\[ + \xi \frac{\partial}{\partial x_{\beta}} \frac{\partial}{\partial x'_{\alpha}} c_{2} + \langle f' \theta \rangle \]

\[ + D^{\beta} (x' - x) \left[ \frac{\partial}{\partial x_{\alpha}} \frac{\partial}{\partial x_{\beta}} + \frac{\partial}{\partial x'_{\alpha}} \frac{\partial}{\partial x'_{\beta}} \right] c_{2} \]

\[ + \xi \frac{\partial}{\partial x'_{\beta}} \frac{\partial}{\partial x_{\alpha}} c_{2} + \langle f' \theta' \rangle \]

\[ = \left\{ D^{\beta} (x - x') \left[ \frac{\partial}{\partial x_{\alpha}} \frac{\partial}{\partial x_{\beta}} + \frac{\partial}{\partial x'_{\alpha}} \frac{\partial}{\partial x'_{\beta}} + \frac{\partial}{\partial x'_{\alpha}} \frac{\partial}{\partial x_{\beta}} + \frac{\partial}{\partial x_{\alpha}} \frac{\partial}{\partial x'_{\beta}} \right] \right. \]

\[ + \xi \left( \frac{\partial}{\partial x_{\beta}} \frac{\partial}{\partial x_{\alpha}} + \frac{\partial}{\partial x'_{\beta}} \frac{\partial}{\partial x'_{\alpha}} \right) \right\} c_{2} + \langle f' \theta + f' \theta' \rangle \]
\[ \begin{align*}
\frac{\partial C_2}{\partial t} &= M_2 C_2 + \mathcal{F}_2. \\
\text{In the steady state, } \frac{\partial}{\partial t} C_2 &= 0 \\
\Rightarrow \quad M_2 C_2 &= \mathcal{F}_2 \\
\Rightarrow \quad C_2 &= M_2^{-1} \mathcal{F}_2
\end{align*} \]

Later on about \( \kappa \).

Now let us look at higher order correlation functions.

\( C_{2N}(x_1, \ldots, x_{2N}) = \langle \theta(x_1) \theta(x_2) \ldots \theta(x_{2N}) \rangle \)

\( \partial_t C_{2N} = \sum_{j=1}^{2N} \langle \theta(x_j) \left[ \partial_t \theta(x_j) \right] \ldots \theta(x_{2N}) \rangle \)

\[ \begin{align*}
\partial_t C_{2N} &= \sum_{j=1}^{2N} \langle \theta(x_j) \left[ \partial_t \theta(x_j) \right] \ldots \theta(x_{2N}) \rangle \\
&= \sum_{j=1}^{2N} \langle \theta(x_j) \left[ -u_\alpha(x_j) \frac{\partial}{\partial x_j} + \kappa \frac{\partial^2}{\partial x_j^2} \right] \theta(x_j) + \mathcal{F}(x_j) \rangle \ldots \rangle \\
&= \partial \varepsilon \sum_{j=1}^{2N} \frac{\partial}{\partial x_j} C_{2N} + \sum_{j, k} \langle \mathcal{G}(x_j - x_k) C_{2N-2}(x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, x_{2N}) \rangle
\end{align*} \]
\[ \partial_t C_{2N} = \left[ \sum \partial_\alpha (x_j - x_k) \partial^{x_j} \partial^{x_k} + \lambda \sum \partial_{\beta \rho} \right] C_{2N} \]

\[ + \sum_{j, k} \theta(x_j - x_k) C_{2N-2} \]

\[ \partial_t C_{2N} = M_{2N} C_{2N} + \Phi_2 C_{2N-2} \]
\[ \partial_t C_{2N} = M_{2N} C_{2N} + \Phi_2 C_{2N-2}. \]

At steady state: \[ M_{2N} C_{2N} = \Phi_2 C_{2N-2}. \]

What is the naive scaling dimension of \( M_{2N} \)?

\[ M_{2N} = \sum_{i,j} \Theta_i (x_i - x_k) \frac{x_i}{r} \frac{x_j}{r} + \rho \sum \Theta_\rho \]

Assume zero diffusivity.

Then

\[ M_{2N} \sim \frac{r^3}{r^2} \]

\[ M_{2N} \sim r^{3-2} M_{2N-2} \]

\[ \Rightarrow \]

\[ C_{2N} \sim r^{2-3} \Phi_2 C_{2N-2}. \]

\[ \Rightarrow \]

\[ C_2 \sim r^{2-5} \]

\[ C_{2N} \sim r^{2N(2-5)} \]

which is just dimensional scaling.
How can this "power-counting" go wrong?

(i) The inverse of the operator \( M_{2N} \) is an integral operator. If the integrals do not converge then the different cut-off of the integral might play a role and we might get something other than dimensional contribution.

(ii) If there are zero modes

let us look at convergence properties:

For the second order structure function.

To do this carefully let us first look at \( D^{1/2} \)

\[
D^{1/2}(r) = \int D^{1/2}(k) e^{i k \cdot r} d^3k
\]

\[
= \int \left( \frac{\delta^{1/2} - \frac{k k^2}{k^2}}{k^2 + k_0^2 \left( \frac{d + 1}{2} \right)} \right) e^{-(k^2)} e^{i k \cdot r} d^3k
\]

\[
= D_0 \delta^{1/2} - \frac{1}{2} D^{1/2}(r)
\]

where \( D_0 \approx 0.5 \).
\[
\lim_{\eta \to 0} \alpha^\eta(x) = D_\alpha \xi \left[ (d - 1 + \varepsilon) \alpha^\eta - \varepsilon \frac{\alpha^\eta}{r^2} \right]
\]

What does this imply for the velocity field?

1. If we first take the limit of \( \eta \to 0 \) then the velocity field is no longer differentiable in a Taylor series for \( \varepsilon < 2 \). This is called "rough". Velocities are then Hölder continuous with exponent \( \frac{\varepsilon}{2} - 0 \).

A function is called Hölder continuous with exponent \( \alpha \) when

\[
\lim_{x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x} = f'(x)
\]

A function \( f(x) \) is called Lipschitz continuous if

\[
f(x) - f(y) \leq L |x - y| \quad \text{for} \quad x \to y,
\]

where \( L \) is a constant.

For non-Lipschitz functions, if a function satisfies

\[
f(x) - f(y) \leq C |x - y|^\alpha \quad \text{as} \quad x \to y,
\]

Then the function is called "\( \alpha \)-Hölder".
Thus in the when we are looking for multiscaling of the Kraichnan model we are looking at a system where the advection is by velocities which are not H^1 only \( \frac{3}{2} \) Hölder. Except when \( s = 2 \) when the velocity is smooth.

With this form for \( D(\tau) \); we have

\[
M_2(\rho) = \sum_{i=1,2} \left[ \frac{\partial}\partial \beta_i \right]^{3/2} \beta_i \beta_j + \sum_{i=1,2} \beta_i \beta_i
\]

For \( i \neq j \)

\[
M_2(\rho) = \beta_i \beta_j \beta_i \beta_j + \text{other terms.}
\]

If you now take the limit of \( L \to \infty \), or \( k \to 0 \)

so the \( i = j \) part of \( M_2(\tau) \) blows up. The way to tackle this problem is to use the structure function instead of the correlation function.
In steady state $C_2$ satisfies the equation:

$$-M_2 C_2 = \mathcal{F}_2$$

$$\sum_{i,j=1}^{2} D(x_i - x_j) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} + \sum_{i=1}^{2} \frac{\partial}{\partial x_i} \int \left[ C_2(x_i, x_j) = \mathcal{F}_2(x_i, x_j) \right]$$

- Translational invariance $C_2(x_{i} - x_{j}) = C_2(x_{i} - x_{j})$

- Isotropy $C_2(x_{i} - x_{j}) = C_2(|x_{i} - x_{j}|) = C_2(r)$

$$- r^{1-d} \partial_{r} \left[ (d-1) \delta_{1} + d-1 + \frac{2}{r} r^{d-1} \right] \partial_{r} C_2(r) = \Theta \mathcal{F}_2 \left( \frac{r}{L} \right)$$

$$\Rightarrow C_2(r) = \frac{1}{(d-1) \delta_{1}} \int_{r}^{\infty} \frac{x^{1-d} dx}{x^{3} + \frac{2}{L} x^{2} + \frac{2}{L}} \left[ \mathcal{F}_2 \left( \frac{x}{L} \right) \right] r^{d-1} dr$$

with the boundary condition $C_2(0) = \text{White}$

$C_2(\infty)$ is zero.
Let us look at the consequences:

- For fixed $L$, and fixed $\eta$

  with $\eta \ll r \ll L$

  \[ c_2(r) \approx \frac{\Phi(0) L}{(2-3)(4+3-2)(d-1) D_1} - \frac{\Phi(0)}{(2-3) d (d-1) D_1}. \]

  \[ \Rightarrow \quad S_2(r) \approx \frac{2}{(2-3) d (d-1) D_1} \Phi(0) r^{2-3} \]

  \[ S_2 = 2-3. \]

- For $r \ll \eta$

  \[ S_2(r) \approx \frac{1}{2 \pi d} \Phi(0) r^2. \]
Then the only way multiscaling can appear is by zero modes.

\[ M_{2N} C_{2N} = - \Phi_2 C_{2N-2} \]

Only the "zero modes" of $M_{2N}$ can escape dimensional prediction. But to see multiscaling the following should also be true.

1. The "zero modes" should have scaling properties (e.g. constant are not useful).

2. The "zero modes" should satisfy the right boundary condition.

3. Each order should have its own zero modes which cannot be constructed out of the zero modes of smaller orders.

4. It turns out that

4. It turns out that

4. The scaling exponents of the zero modes should be less than the dimensional exponent.
Zero modes and multiscaling

\[ z_2 = 2 \]

\[ z_{kn} = N(2 - g) - \frac{2N(2N-2)}{(d+2)} g + O(g^2) \]

There are three well behaved limits

\[ g = 0 \]

\[ g = 2 \]

and \( d \to \infty \)

There exists three perturbation expansions from these three directions

\[ J_{2n} = N(2 - g) - \frac{2N(N-1)}{(d+2)} g + O(g^2) \]

( Bernard, Gawedzki, Kupiainen 1996)

\[ J_{2n} = N(2 - g) - 2N(C(N-1)) \frac{g}{d} + O(g^2) \]

( Chertkov & Falkovich 1996)
Particle and fields.

\[ \dot{\mathbf{r}} = \mathbf{u} \]

\[ \mathbf{u}(t_1, \mathbf{r}_0, t_0) = \frac{d\mathbf{R}}{dt} \quad \text{Lagrangian velocity.} \]

\[ \mathbf{u}(t, \mathbf{r}) = \mathbf{u}(t_1, \mathbf{r}_0, t_0) \quad \text{when} \quad \mathbf{r}_0 + R = \mathbf{r} \]

\[ \frac{d\mathbf{R}}{dt} = \mathbf{u}(t, \mathbf{r}) \]

\[ \text{Therefore prove} \]

\[ d\mathbf{R} = \mathbf{u}(t, \mathbf{r}) dt \]

\[ \approx \text{Material RT} \]

\[ \text{If } R \]

**Statement.** If \( R(t_1, \mathbf{r}_0) \) solves eqn. (1) with the initial condition that \( R(t_0, t_0, \mathbf{r}_0) = \mathbf{r}_0 \) then

\[ \Theta(t, \mathbf{r}) = \int S \left[ \mathbf{r} - R(t_1, \mathbf{r}_0) \right] \Theta(t_0, \mathbf{r}_0) d\mathbf{r}_0 \]

solves the equation

\[ \partial_t \Theta + (\mathbf{u} \cdot \nabla) \Theta = 0. \]
Let us first prove this in one dimension.

\[ \frac{\partial u(x,t)}{\partial t} = u(x,t) \]

Let \( \theta(t, x_0, t_0) \) be a solution of the preceding equation and \( \Theta(t) \) meaning of the above statement is to be understood in the "weak" sense; or in the sense of "distributions." i.e. for \( \psi(x) \) in a class of \( \Theta \) in a class of test functions

\[ \int \psi(x) \theta(t, x) \, dx = \int \delta[x-x_0] \psi(x) \theta(t_0, x_0) \, dx_0 \, dx \]

and this solves the equation

\[ \int \psi(x) \partial_t \theta \, dx + \int \psi(x) (\mu \cdot \nabla) \theta \, dx = 0 \]
Weak solution

A careful way of saying that we are allowing
and "non-functions" like \( \delta \) functions to be solutions to the
differential equations. Let us look at more familiar example.

Poisson eqn for a point mass is

\[ \nabla^2 \phi(x) = -\frac{\delta(x)}{\varepsilon_0} \]

For a point mass charge we know what the solution is:

\[ \phi(x) = q \left( \frac{1}{r} - \frac{x}{r^2} \right) \]

\[ \phi(x) = \frac{q}{4\pi \varepsilon_0} + \frac{q}{4\pi \varepsilon_0} \frac{1}{r} \]

or

\[ \frac{1}{r} = -4\pi \delta(x) \]

This is to be interpreted in the "weak" sense. i.e.

\[ \int \psi(x) \nabla^2 \phi(x) \, dx = -\int \frac{\delta(x)}{\varepsilon_0} \, dx \psi(x) \, dx \]
\[ \frac{\partial}{\partial t} \left( \psi(x) \theta(x, t) \right) dx = \int \psi(x) \frac{\partial}{\partial t} \delta \left( x - a(t, y_0) \right) \theta(y, t_0) dy \, dx \\
= -\int \psi(x) \theta(x, t_0) \frac{d}{dt} \int \psi(x) \delta \left( x - a(t, y_0) \right) \theta(y, t_0) dy \, dx \\
= -\int \psi(x) \theta(x, t_0) \frac{d}{dt} \left[ \psi(x) \theta(x, t) \right] dx \\
= -\int \psi(x) \frac{d}{dx} \left[ \psi(x) \theta(x, t) \right] \theta(x, t) dx \\
= -\int \psi(x) \frac{d}{dx} \theta(x, t) \theta(x, t) dx \\
\Rightarrow \frac{\partial}{\partial t} \theta = -u(x) \frac{d}{dx} \theta \]

in several dimensions: \[ \frac{\partial}{\partial t} \theta = -(u \cdot \nabla) \theta \]
Statement

If we now consider $R(t, t_0, r_0)$ to be a solution of the equation

$$\frac{dR}{dt} = u(x, t) + \sqrt{2 \pi} \tilde{\xi} \gamma,$$

where $\gamma$ is white noise, then

$$\theta(t, \gamma) = \langle \int \delta \left[ \gamma - R(t, t_0, r_0) \right] \theta(t_0, r_0) \, dr_0 \rangle$$

over $\gamma$.

satisfies the following PDE

$$\partial_t \theta + (\mathbf{u} \cdot \nabla) \theta = \mathbb{R}^2 \nabla^2 \theta.$$  

Before we proceed for a proof, we have to understand the notion of SDE.

$$dR = u(y, t) \, dt + \sqrt{2 \pi} \, d\beta$$

where $d\beta = \int_t^{t+dt} \gamma \, dt$.

$$W^*_{\phi}(t, t_0, r_0) = \frac{\partial}{\partial r_0} R^*_{\phi}(t, t_0, r_0)$$

$$B(t, \gamma) = \int \delta \left[ \gamma - \mathbf{B} \right] \mathbf{B}(t_0, r_0) \, dr_0$$

$$\frac{\partial B}{\partial t} + (\mathbf{u} \cdot \nabla) B + B \left( \mathbf{v} \cdot \nabla \right) - (\nabla \cdot B) u - \mathbb{R}^2 \nabla^2 B = 0.$$
Stochastic Differential Equations

So first consider the SDE in one dimension

\[ da = u \, dt + \sqrt{2 \sigma^2} \, d\beta. \]

Consider

\[ d[f(a)]_{t=0} = f[a(t+\delta t)] - f[a(t)] \]

\[ = f[a + da] - f[a] \]

\[ = f(a) + \frac{df}{da} \cdot da + \frac{1}{2} \frac{d^2 f}{da^2} \cdot (da)^2 + O(da^3) \]

Then

\[ = \frac{df}{da} \cdot da + \frac{1}{2} \frac{d^2 f}{da^2} \cdot (da)^2 + O(da^3) \]

\[ da^2 = u^2 \, dt^2 + 2u \, dt \cdot d\beta \sqrt{2\sigma^2} + 2\sigma^2 \, d\beta^2 \]

\[ \beta^2 = \int \int \gamma(t_1) \, dt_1 \gamma(t_2) \, dt_2 \]

\[ \langle d\beta \rangle = \int \int \langle \gamma(t_1) \gamma(t_2) \rangle \, dt_1 \, dt_2 = dt \]

\[ d[f(a)]_{t=0} = \frac{df}{da} \left( u \, dt + \sqrt{2\sigma^2} \, d\beta \right) + \sigma \frac{d^2 f}{da^2} \, dt \]

\[ d[f(a)]_{\text{Stratanovich}} = \frac{df}{da} \left( u \, dt + \sqrt{2\sigma^2} \, d\beta \right) \]
\textbf{proof}

\[
d\alpha = \sqrt{2\pi}\, d\beta.
\]

\[
\Theta(x, t) = \int \delta \left[ x - a(t; y, t_0) \right] \, d\beta \, \Theta(y, t_0) \, dy
\]

or

\[
\int \psi(dy) \Theta(x, t) \, dx = \int \psi(x) \delta \left[ x - a(t) \right] \Theta(y, t_0) \, dy \, dx
\]

\[
\int \psi \frac{d\Theta}{dt} \, dx = \int \psi(x) \delta' \left[ x - a \right] \frac{da}{dt} \Theta(y, t_0) \, dy
\]

\[
= \frac{d}{dt} \int \psi(x) \Theta(y) \, dy
\]

\[
= \frac{d}{dt} \int x \frac{d\psi}{da} \Theta(y) \, dy
\]

\[
= \int x \frac{d}{da} \Theta(y) \, dy
\]

\[
= \int x \Theta(y) \frac{d}{da} \psi \, dy
\]

\[
= \int x \Theta(y) \frac{d^2}{da^2} \psi \, dy
\]

\[
= \int x \Theta(y) \frac{d^2}{da^2} \int \psi(x) \delta(x - a) \, dx \, dy
\]

\[
= \int x \Theta(y) \psi(x) \frac{d^2}{da^2} \delta(x - a) \, dx \, dy
\]

\[
= \int x \psi(x) \frac{d^2}{da^2} \delta(x - a) \Theta(y) \, dx \, dy
\]

\[
= \int x \psi(x) \frac{d^2}{da^2} \delta(x - a) \Theta(y) \, dx \, dy
\]

\[
= \int x \psi(x) \frac{d^2}{da^2} \Theta(x) \, dx \quad \text{Q.E.D.}
\]
Particle picture

\[ \text{d} \mathbf{R} = \mathbf{u}(\mathbf{R}, t) \text{d}t + \sqrt{2\lambda} \text{d} \mathbf{B}(t) \]

For \( \lambda = 0 \), this is just the Langevin eqn.

\[ \Delta \mathbf{R}(t) = \mathbf{R}(t) - \mathbf{R}(0) \]

The probability of having a value \( \Delta \mathbf{R} \), \( P[\Delta \mathbf{R}] \)
satisfies the eqn

\[ \left[ \partial_t - \lambda \nabla^2 \right] P = 0. \]

\[ P(\Delta \mathbf{R}, t) = \frac{1}{(4\pi \lambda t)^{-d/2}} \exp \left[ - \frac{(\Delta \mathbf{R})^2}{4\lambda t} \right] \]

For \( \lambda = 0 \),

\[ \frac{d}{dt} \langle (\Delta \mathbf{R})^2 \rangle = 2 \int_0^t \langle \mathbf{V}(0) \cdot \mathbf{V}(s) \rangle \text{d}s \]

\textbf{Proof}

\[ \frac{\text{d}a}{\text{d}t} = \mathbf{u}(t \mid \mathbf{R}, t_0) \]

\[ a(t) = \int_0^t \mathbf{V}(s) \text{d}s \]

\[ [a(t) - a(0)]^2 = \int \int \mathbf{V}(t_1) \cdot \mathbf{V}(t_2) \text{d}t_1 \text{d}t_2 \]

\[ + \frac{2}{\lambda} (a(t) - a(0)) \cdot \mathbf{u}(0) \]

\[ + 2 \cdot a(0) \int_0^t \mathbf{V}(t_1) \text{d}t_1 \]

\[ + \frac{2}{\lambda} a(0)^2 \]

\[ \Rightarrow \frac{d}{dt} \langle [a(t) - a(0)]^2 \rangle = 2 \int_0^t \langle \mathbf{V}(t) \cdot \mathbf{V}(t_2) \rangle \text{d}t_2 \]

\[ = 2 \int_0^t \langle \mathbf{V}(0) \cdot \mathbf{V}(s) \rangle \text{d}s \]
\[ r = \int_0^\infty \frac{\langle v(0) \cdot v(s) \rangle ds}{\langle v(0) \rangle} \]

- Divergence of \( r \) implies persistence of correlations.

- If the above integral converges,
  \[ \frac{d}{dt} \langle (AR)^2 \rangle = 2 \langle v^2 \rangle r \Rightarrow \langle (AR)^2 \rangle = 2 \langle v^2 \rangle r t \]
  - Diffusion.

- For small time \( t < r \),
  \[ \frac{d}{dt} \langle (AR)^2 \rangle = 2 \langle v^2 \rangle t \Rightarrow (AR)^2 = 2 \langle v^2 \rangle t^2 \]
  - Ballistic.

- For \( t > r \), and for finite \( r \), \( (AR) \) behaves like sum over many random variables. Then \( (AR) \) behaves like Brownian motion in these dimensions.

- Divergence of the integral implies super diffusion.
Two particle separation:

\[ R_{12} = R_1 - R_2 \]

- Smooth velocities. (Batchelor regime)

\[ \frac{dR_{12}}{dt} = U(R_1) - U(R_2) \]

\[
\begin{align*}
\sigma^{r_2} = \frac{1}{3} \sum_i \sigma_i \\
\dot{\sigma}_{12} = \sigma_{12} \gamma \end{align*}
\]

\[ \dot{R}_{12}^R = \frac{d^R R_{12}}{dt} = \sigma^{r_2} R_{12}^R \text{ with } \sigma^{r_2} = \frac{3}{2} \nu \]

\[
\begin{bmatrix}
\dot{R}_{12}^R(t) \\
\end{bmatrix}
= W(t) \begin{bmatrix}
R_{12}^R(0) \\
\end{bmatrix}
\]

First in 1-dimension:

\[ \sigma_{12} = \sigma_{12} \]

\[ \ln \left[ \frac{\sigma_{12}(t)}{\sigma_{12}(0)} \right] = \ln \left[ W(t) \right] = \int_0^t \sigma(s) \, ds \equiv X. \quad X = \sum_{i=1}^N \gamma_i, \quad N = \frac{t}{\tau} \]

For \( t \gg \tau \) the correlation time of \( \sigma \), \( X \) is a sum of many i.i.d random numbers. \( P(X) \) has a large-deviation form.

\[ \langle X \rangle = N \langle \gamma \rangle \text{ grows linearly with time}. \]

Or the interparticle distance grows exponentially with time, with exponent

\[ \lambda = \frac{\langle X \rangle}{t} \text{ - Lyapunov exponent}. \]
In several dimension
\[ w(t) = T \exp \left[ \int_0^t \! \sigma(s) \, ds \right] \]
\[ = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^t \sigma^n(s) \, ds \cdot \int_0^t \sigma^n(s) \, ds \]
\[ \lambda_i = \lim_{t \to \infty} \frac{1}{t} \ln |w(t)| \quad \text{Lyapunov exponents} \]

In compressibility \( \Rightarrow \) \( \det(w) = 1 \), \( \sum \lambda_i = 0 \).

Imples volume is preserved.

Consider \( I = w^T w \).

For delta correlated strain
\[ \langle \delta \eta^{\alpha} \delta \eta^{\beta} \rangle = \frac{3}{2} \delta(t-t') \delta^{\alpha\beta} \]
\[ \langle \frac{\partial u^{\alpha}}{\partial x^i} \frac{\partial u^{\beta}}{\partial x^j} \rangle = \frac{3}{2} \frac{\partial^{i+j} \rho}{\partial x^i \partial x^j} \delta(t-t') \]
\[ \langle u^\alpha \eta^{\beta} \rangle = \frac{3}{2} \frac{\partial \rho}{\partial x^i} \delta(t-t') \delta^{\alpha\beta} \]
\[ \approx \frac{3}{2} \frac{\partial \rho}{\partial x^i} \delta(t-t') \delta^{\alpha\beta} \]