Basic notions and steady flows

In this chapter, we define the subject, derive the equations of motion and describe their fundamental symmetries. We start from hydrostatics where all forces are normal. We then try to consider flows this way as well, neglecting friction. This allows us to understand some features of inertia, most importantly induced mass, but the overall result is a failure to describe a fluid flow past a body. We are then forced to introduce friction and learn how it interacts with inertia, producing real flows. We briefly consider an Aristotelean world where friction dominates. In an opposite limit, we discover that the world with a little friction is very much different from the world with no friction at all.

1.1 Definitions and basic equations

Here we define the notions of fluids and their continuous motion. These definitions are induced by empirically established facts rather than deduced from a set of axioms.

1.1.1 Definitions

We deal with continuous media where matter may be treated as homogeneous in structure down to the smallest portions. The term fluid embraces both liquids and gases and relates to the fact that even though any fluid may resist deformations, that resistance cannot prevent deformation from happening. This is because the resisting force vanishes with the rate of deformation. With patience, anything can be deformed. Therefore, whether one treats the matter as a fluid or a solid depends on the time available for observation. As the prophetess Deborah sang, “The mountains flowed before the Lord” (Judges 5:5). The ratio of the relaxation...
time to the observation time is called the Deborah number. The smaller the number the more fluid the material.

A fluid can be in mechanical equilibrium only if all the mutual forces between two adjacent parts are normal to the common surface. That experimental observation is the basis of hydrostatics. If one applies a force parallel (tangential) to the common surface then the fluid layer on one side of the surface starts sliding over the layer on the other side. Such sliding motion will lead to a friction between layers. For example, if you cease to stir tea in a glass it could come to rest only because of such tangential forces, i.e. friction. Indeed, if the mutual action between the portions on the same radius was wholly normal, i.e. radial, then the conservation of angular momentum about the rotation axis would cause the fluid to rotate forever.

Since tangential forces are absent at rest or for a uniform flow, it is natural to consider first the flows where such forces are small and can be neglected. Therefore, a natural first step out of hydrostatics into hydrodynamics is to restrict ourselves to purely normal forces, assuming small velocity gradients (whether such a step makes sense at all and how long such approximation may last remains to be seen). Moreover, the intensity of a normal force per unit area does not depend on the direction in a fluid (Pascal’s law, see Exercise 1.1). We thus characterize the internal force (or stress) in a fluid by a single scalar function \( p(\mathbf{r}, t) \) called pressure, which is the force per unit area. From the viewpoint of the internal state of the matter, pressure is a macroscopic (thermodynamic) variable. Microscopically, we assume every portion of the fluid to be in thermal equilibrium. In this case, the internal state of the fluid is described completely by two variables, so one needs a second thermodynamical quantity. We shall usually use the density \( \rho(\mathbf{r}, t) \), in addition to the pressure.

What analytic properties of the velocity field \( \mathbf{v}(\mathbf{r}, t) \) do we need to presume? We suppose the velocity to be finite and a continuous function of \( \mathbf{r} \). In addition, we suppose the first spatial derivatives to be everywhere finite. That makes the motion continuous, i.e. trajectories of the fluid particles do not cross. The equation for the distance \( \delta \mathbf{r} \) between two close fluid particles is \( \frac{d\delta \mathbf{r}}{dt} = \delta \mathbf{v} \) so, mathematically speaking, the finiteness of \( \nabla \mathbf{v} \) is the Lipschitz condition for this equation to have a unique solution (a simple example of non-unique solutions for non-Lipschitz equation is \( \frac{dx}{dt} = |x|^{1-\alpha} \) with two solutions, \( x(t) = (\alpha t)^{1/\alpha} \) and \( x(t) = 0 \), starting from zero for \( \alpha > 0 \)). For a continuous motion, any surface moving with the fluid completely separates matter on the two sides of it. We don’t yet know when exactly the continuity assumption is consistent with the equations of the fluid motion. Whether velocity derivatives may turn into infinity after a finite time is a subject of active research for an incompressible viscous
fluid (and a subject of a one-million-dollar Clay prize). We shall see that a compressible inviscid flow generally develops discontinuities, called shocks.

1.1.2 Equations of motion for an ideal fluid

The Euler equation. The force acting on any fluid volume is equal to the pressure integral over the surface: $-\oint p \, df$. The surface area element $df$ is a vector directed as outward normal:

Let us transform the surface integral into the volume one: $-\int \nabla p \, dV$. The force acting on a unit volume is thus $-\nabla p$. That would be wrong, however, to assume that this force is the time derivative of the momentum $\rho v$ of this volume. To write the second law of Newton, we need to single out a fixed body of fluid. An infinitesimal such body is called fluid particle and it always contains the same mass, which we assume unity. Then the force per unit mass, $\nabla p/\rho$, must be equal to the acceleration $dv/dt$:

$$\frac{dv}{dt} = -\frac{\nabla p}{\rho}.$$  

The acceleration $dv/dt$ is not the rate of change of the fluid velocity at a fixed point in space but the rate of change of the velocity of a given fluid particle as it moves about in space. One uses the chain rule of differentiation to express this (substantial or material) derivative in terms of quantities referring to points fixed in space. During the time $dt$ the fluid particle changes its velocity by $dv$ (which is composed of two parts, temporal and spatial):

$$dv = \frac{\partial v}{\partial t} + (dr \cdot \nabla)v = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}. \quad (1.1)$$

It is the change in the fixed point plus the difference at two points $dr$ apart, where $dr = vdt$ is the distance moved by the fluid particle during $dt$ due to inertia. Dividing (1.1) by $dt$ we obtain the substantial derivative as a local derivative plus a convective derivative:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + (v \cdot \nabla)v.$$
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![Figure 1.1](image)

Figure 1.1 The radial pressure gradient is normal to circular surfaces and cannot change the moment of momentum of the fluid inside or outside the surface; it changes the direction of velocity $v$ but not its modulus.

We see that even when the flow is steady, $\partial v/\partial t = 0$, the acceleration is non-zero as long as $(v \cdot \nabla)v \neq 0$, that is if the velocity field changes in space along itself. Any function $F(r(t), t)$, like fluid temperature, varies for a moving particle in the same way, according to the chain rule of differentiation:

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + (v \cdot \nabla)F.$$

Writing now the second law of Newton for a unit mass of a fluid, we come to the equation derived by Euler (Berlin, 1757; Petersburg, 1759):

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p/\rho.$$  \hspace{1cm} (1.2)

Before Euler, the acceleration of a fluid had been considered as due to the difference of the pressure exerted by the enclosing walls. Euler introduced the pressure field inside the fluid. For example, for the steadily rotating fluid shown in Figure 1.1, the acceleration vector $(v \cdot \nabla)v$ has a non-zero radial component $v^2/r$. The radial acceleration times the density gives the radial pressure gradient:

$$\frac{dp}{dr} = \rho \frac{v^2}{r}.$$  \hspace{1cm} (1.3)

We can also add an external body force per unit mass (for gravity $f = g$):

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p/\rho + f.$$  \hspace{1cm} (1.3)

The term $(v \cdot \nabla)v$ describes inertia and makes (1.3) non-linear.

**Continuity equation.** This expresses conservation of mass. If $Q$ is the volume of a moving element then $d\rho Q/\partial t = 0$, that is

$$Q \frac{d\rho}{dt} + \rho \frac{dQ}{dt} = 0.$$ \hspace{1cm} (1.4)


The volume change can be expressed via $v(r, t)$.

The horizontal velocity of the point B relative to the point A is $\delta x \partial v_x / \partial x$.

After the time interval $\delta t$, the length of the edge $AB$ is $\delta x(1 + \delta t \partial v_x / \partial x)$. Overall, after $\delta t$, one has the volume change

$$dQ = dt \delta x \delta y \delta z \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = dt Q \text{div} v = dt \frac{dQ}{dt}.$$  

Substituting that into (1.4) and cancelling (arbitrary) $Q$ we obtain the continuity equation

$$\frac{d\rho}{dt} + \rho \text{div} v = \frac{\partial \rho}{\partial t} + (\mathbf{v} \cdot \nabla) \rho + \rho \text{div} v = \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0. \quad (1.5)$$

The last equation is almost obvious since for any fixed volume of space the decrease of the total mass inside, $- \int (\partial \rho / \partial t) \, dV$, is equal to the flux $\oint \rho \mathbf{v} \cdot d\mathbf{f} = \int \text{div}(\rho \mathbf{v})dV$.

**Entropy equation.** We now have four equations (1.3, 1.5) for five quantities $p, \rho, v_x, v_y, v_z$, so we need one extra equation. In deriving (1.3, 1.5) we have taken no account of energy dissipation, thus neglecting internal friction (viscosity) and heat exchange. A fluid without viscosity and thermal conductivity is called ideal. The motion of an ideal fluid is adiabatic, that is the entropy of any fluid particle remains constant: $\frac{ds}{dt} = 0$, where $s$ is the entropy per unit mass. We can turn this equation into a continuity equation for the entropy density in space

$$\frac{\partial (\rho s)}{\partial t} + \text{div}(\rho s \mathbf{v}) = 0. \quad (1.6)$$

Since entropy is a function of pressure and density then (1.6) is the needed extra relation between velocity, pressure and density. Different media differ by the form of the function $s(P, \rho)$.
Boundary conditions. At the boundaries of the fluid, the continuity equation (1.5) is replaced by the boundary conditions:

1. On a fixed boundary, \( v_n = 0 \);
2. On a moving boundary between two immiscible fluids, \( p_1 = p_2 \) and \( v_{n1} = v_{n2} \).

These are particular cases of the general surface condition. Let \( F(\mathbf{r}, t) = 0 \) be the equation of the bounding surface. An absence of any fluid flow across the surface requires

\[
\frac{dF}{dt} = \frac{\partial F}{\partial t} + (\mathbf{v} \cdot \nabla) F = 0,
\]

which means, as we now know, the zero rate of \( F \) variation for a fluid particle. For a stationary boundary, \( \frac{\partial F}{\partial t} = 0 \) and \( \mathbf{v} \perp \nabla F \Rightarrow v_n = 0 \).

1.1.3 Hydrostatics

A necessary and sufficient condition for fluid to be in a mechanical equilibrium follows from (1.3):

\[
\nabla p = \rho \mathbf{f}.
\]

Not every distribution of \( \rho(\mathbf{r}) \) could be in equilibrium since \( \rho(\mathbf{r}) \mathbf{f}(\mathbf{r}) \) is not necessarily a gradient. If the force is potential, \( \mathbf{f} = -\nabla \phi \), then taking the curl of (1.7) we get

\[
\nabla \rho \times \nabla \phi = 0.
\]

This means that the gradients of \( \rho \) and \( \phi \) are parallel and their level surfaces coincide in equilibrium. The best-known example is gravity with \( \phi = gz \) and \( \partial p/\partial z = -\rho g \). For an incompressible fluid, it gives

\[
p(z) = p(0) - \rho g z.
\]

For an ideal gas under a homogeneous temperature, which has \( p = \rho T/m \), one gets

\[
\frac{dp}{dz} = -\frac{\rho gm}{T} \Rightarrow p(z) = p(0) \exp(-mgz/T).
\]

For air at 0°C, \( T/mg \simeq 8 \) km. The Earth’s atmosphere is described by neither a linear nor an exponential law because of an inhomogeneous temperature (Figure 1.2). Assuming a linear temperature decay, \( T(z) = T_0 - \alpha z \), one obtains
1.1 Definitions and basic equations

Figure 1.2 Pressure–height dependence for an incompressible fluid (broken line), isothermal gas (dotted line) and a real atmosphere (solid line).

a better approximation:

\[
\frac{dp}{dz} = -\rho g = -\frac{pmg}{T_0 - \alpha z},
\]

\[
p(z) = p(0)(1 - \alpha z/T_0)^{mE/\alpha},
\]

which can be used not far from the surface with \(\alpha \approx 6.5^\circ\text{C}\text{km}^{-1}\).

Under gravity, density depends only on the distance from the Earth center (or locally on on the vertical coordinate \(z\)) in a mechanical equilibrium. According to \(dp/dz = -\rho g\), the pressure also depends only on \(z\). Pressure and density determine temperature, which must then also be independent of the horizontal coordinates. Different temperatures at the same height, in particular non-uniform temperature of the Earth surface, necessarily produce fluid motion, which is why winds blow in the atmosphere and currents flow in the ocean. Another source of atmospheric flows is thermal convection due to a negative vertical temperature gradient. Let us derive the stability criterion for a fluid with a vertical profile \(T(z)\). If a fluid element is shifted up adiabatically from \(z\) by \(dz\), it keeps its entropy \(s(z)\) but acquires the pressure \(p' = p(z + dz)\) so its new density is \(\rho(s, p')\). For stability, this density must exceed the density of the displaced air at the height \(z + dz\), which has the same pressure but different entropy \(s' = s(z + dz)\). The condition for stability of the stratification is as follows:

\[
\rho(p', s) > \rho(p', s') \Rightarrow \left( \frac{\partial \rho}{\partial s} \right)_p \frac{ds}{dz} < 0.
\]

Entropy usually increases under expansion, \((\partial \rho/\partial s)_p < 0\), and for stability we must require \(ds/dz > 0\). Entropy depends on \(p, T\) which both decay with the height. Entropy decreases with cooling yet increases when \(P\) decreases. To see
which effect wins we compute:

$$\frac{ds}{dz} = \left( \frac{\partial s}{\partial T} \right)_p \frac{dT}{dz} + \left( \frac{\partial s}{\partial p} \right)_T \frac{dp}{dz} = \left( \frac{\partial V}{\partial T} \right)_p g > 0. \quad (1.8)$$

Here we used specific volume $V = 1/\rho$. For an ideal gas the coefficient of the thermal expansion gives $\left( \frac{\partial V}{\partial T} \right)_p = V/T$ and we end up with

$$\frac{g}{c_p} > -\frac{dT}{dz}. \quad (1.9)$$

Indeed, stability requires that the gain in potential energy $gdz$ must exceed the decrease in thermal energy $c_p dT$. For the Earth’s atmosphere, $c_p \sim 10^3 J/kg^{-1} K^{-1}$ and the convection threshold is $10^3$ Ckm$^{-1}$. The average gradient is $6.5^\circ Ckm^{-1}$, that is the entropy decreases with the height and the atmosphere is globally stable. However, local gradients vary very much depending on ground albedo, evaporation etc, so that the atmosphere is often locally unstable with respect to thermal convection. The human body always excites convection in room-temperature air.$^2$

Temperature decays with height only in the troposphere that is until about $-50^\circ$ at 10-12 km, it is then constant up to about 35 km so that the pressure decays exponentially, eventually it grows in the stratosphere until about 0$^\circ$ at 50 km. Looking down from the plane flying above 10 km one often sees flat cloud top, particularly so-called anvil clouds, which is exactly where unstable air stratification below turns into stable above.

The convection stability argument applied to an incompressible fluid rotating with the angular velocity $\Omega(r)$ gives the Rayleigh’s stability criterion, $d(r^2 \Omega)^2/dr > 0$, which states that the angular momentum of the fluid $L = r^2 |\Omega|$ must increase with the distance $r$ from the rotation axis.$^3$ Indeed, if a fluid element is shifted from $r$ to $r'$ it keeps its angular momentum $L(r)$, so that the local pressure gradient $dp/dr = \rho r^2 \Omega^2(r')$ must overcome the centrifugal force $\rho r'(L^2 r^4/r^4)$.

### 1.1.4 Isentropic motion

The simplest motion corresponds to constant $s$ and allows for a substantial simplification of the Euler equation. Indeed, it would be convenient to represent $\nabla p/\rho$ as a gradient of some function. For this end, we need a function that depends on $p, s$, so that at $s = \text{const.}$ its differential is expressed solely via $dp$. There exists the thermodynamic potential called *enthalpy*, defined as $W = E + pV$ per unit mass ($E$ is the internal energy of the fluid). For our purposes, it is
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enough to remember from thermodynamics the single relation \( dE = T\, ds - p\, dV \) so that \( dW = T\, ds + V\, d\rho \) (one can also show that \( W = \partial(E\rho)/\partial\rho \)). Since \( s = \text{const.} \) for an isentropic motion and \( V = \rho^{-1} \) for a unit mass, \( dW = d\rho/\rho \) and, without body forces one has

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla W. \tag{1.10}
\]

Such a gradient form will be used extensively for obtaining conservation laws, integral relations, etc. For example, we can use the vector identity \( A \times (\nabla \times B) = A \cdot (\nabla B) - (A \cdot \nabla)B \) to represent

\[
(v \cdot \nabla)v = \nabla v^2/2 - v \times (\nabla \times v),
\]

and get

\[
\frac{\partial v}{\partial t} = v \times (\nabla \times v) - \nabla(W + v^2/2). \tag{1.11}
\]

The first term on the right-hand side is perpendicular to the velocity. To project (1.11) along the velocity and get rid of this term, we define a streamline as a line whose tangent is everywhere parallel to the instantaneous velocity. The streamlines are then determined by the relations

\[
\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}.
\]

Note that for time-dependent flows streamlines are different from particle trajectories: tangents to streamlines give velocities at a given time while tangents to trajectories give velocities at subsequent times. One records streamlines experimentally by seeding fluids with light-scattering particles; each particle produces a short trace on a short-exposure photograph, and the length and orientation of the trace indicates the magnitude and direction of the velocity. Streamlines can intersect only at a point of zero velocity called the stagnation point.

Let us now consider a steady flow, assuming \( \partial v/\partial t = 0 \), and take the component of (1.11) along the velocity at a point:

\[
\frac{\partial}{\partial t}(W + v^2/2) = 0. \tag{1.12}
\]

We see that \( W + v^2/2 = E + p/\rho + v^2/2 \) is constant along any given streamline, but may be different for different streamlines (Bernoulli, 1738). Bernoulli theorem, of course, is a particular case of energy conservation. The change of
the total energy density $\rho_1 E_1 + \rho_1 v_1^2/2 - \rho_2 E_2 - \rho_2 v_2^2/2$ is not zero along the streamline but is equal to $P_2 - P_1$ which is the work done. This is the reason $W$ rather $E$ enters the conservation law, as also discussed after (1.18). Alternatively, one may say that $W$ is a potential energy of a fluid particle, see (1.41) below. In a gravity field,

$$W + gz + v^2/2 = \text{const.}$$  \hspace{1cm} (1.13)

Without much exaggeration, one can say that most fluid-mechanics estimates use (1.12) or (1.23). Let us consider several applications of this useful relation.

Imagine that our spaceship suffered meteorite attack that left holes in the walls of the cabin and the tank with liquid fuel. We need to estimate how fast we loose air from the cabin and fuel from the tank. Since there is vacuum outside, we can neglect thermal exchange and consider both flows isentropic. Liquid could be treated as incompressible, its internal energy $E$ is then constant without any external force. Bernoulli theorem then gives the limiting velocity with which such a liquid escapes from a large reservoir into vacuum:

$$v = \sqrt{2p_0/\rho}.$$  

For water ($\rho = 10^3 \text{kg m}^{-3}$) at atmospheric pressure ($p_0 = 10^5 \text{N m}^{-2}$) one gets $v = \sqrt{200} \approx 14 \text{ m s}^{-1}$.

For a gas, pressure drop must be accompanied by density change. The adiabatic law, $p/p_0 = (\rho/\rho_0)^\gamma$, gives the enthalpy as:

$$W = \int \frac{dp}{\rho} = \frac{\gamma p}{(\gamma - 1)\rho}.$$  

The limiting velocity for the escape into vacuum can again be found from Bernoulli theorem:

$$\frac{\gamma p_0}{(\gamma - 1)\rho} = \frac{v^2}{2} \Rightarrow v = \sqrt{\frac{2\gamma p_0}{(\gamma - 1)\rho}}.$$  

The velocity is $\sqrt{\gamma/(\gamma - 1)}$ times larger than for an incompressible fluid which corresponds to the limit $\gamma \gg 1$. The gas flows faster because the internal energy of the gas decreases as it flows, thus increasing the kinetic energy. We conclude that a meteorite-damaged spaceship loses the air from the cabin faster than the liquid fuel from the tank. We shall see later that $(\partial P/\partial \rho)_s = \gamma P/\rho$ is the sound velocity squared, $c^2$, so that $v = c\sqrt{2/(\gamma - 1)}$. For an ideal gas with $n$ internal degrees of freedom, $W = E + p/\rho = nT/2m + T/m$ so that $\gamma = (2+n)/n$.  

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For bi-atomic molecules $n = 5$ (3 translations and 2 rotations) at not very high temperature, when vibrations are not excited.

Another frequent occurrence is efflux from a small orifice under the action of gravity. Supposing the external pressure to be the same at the horizontal surface and at the orifice, we apply the Bernoulli relation to the streamline which originates at the upper surface with almost zero velocity and exits with velocity $v = \sqrt{2gh}$ (Torricelli, 1643). The Torricelli formula is not of much use practically to calculate the rate of discharge, which in reality is not equal to the orifice area times $\sqrt{2gh}$, the fact known to wine merchants long before physicists. Indeed, streamlines converge from all sides towards the orifice so that the jet continues to converge for a while after coming out (Figure 1.3). Moreover, the converging motion makes the pressure in the interior of the jet somewhat greater than that at the surface (as is clear from the curvature of streamlines) so that the velocity in the interior is somewhat less than $\sqrt{2gh}$. The experiment shows that contraction ceases and the jet becomes cylindrical at a short distance beyond the orifice. This point is called ‘vena contracta’ and the ratio of the jet area there to the orifice area is called the coefficient of contraction. The estimate for the discharge rate is $\sqrt{2gh}$ times the orifice area times the coefficient of contraction. For a round hole in a thin wall, the coefficient of contraction is experimentally found to be 0.62. Exercise 1.3 presents a particular case where the coefficient of contraction can be found exactly.

The Bernoulli relation is also used in different devices that measure the flow velocity. Probably, the simplest such device is the Pitot tube shown in Figure 1.4. It is open at both ends with the horizontal arm facing upstream. Since the liquid does not move inside the tube then the velocity is zero at the point labelled B. On the one hand, the pressure difference at two points on the same streamline can be expressed via the velocity at A: $P_B - P_A = \rho v^2/2$. On the other hand,
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Figure 1.4 Pitot tube, which determines the velocity $v$ at the point $A$ by measuring the height $h$.

It is expressed via the height $h$ by which liquid rises above the surface in the vertical arm of the tube: $P_B - P_A = \rho gh$. That gives $v^2 = 2gh$.

One may wonder why the Earth atmosphere is not isentropic as remarked in the previous section. Rising water vapour condenses and releases latent heat, making the mean rate of temperature decrease lower than adiabatic.

1.2 Conservation laws and potential flows

In this section we deduce the conservation laws and their straightforward consequences from the equations of motion.

Symmetries and conservation laws. The equations of ideal hydrodynamics (1.3, 1.5, 1.6) express, respectively, the conservation laws of the momentum, mass and entropy. They are invariant with respect to space translations (which brings momentum conservation), and time translations (which brings energy conservation, described in the next subsection). The equations are time-reversible that is invariant with respect to the transformation $t \rightarrow -t$ and $\mathbf{v} \rightarrow -\mathbf{v}$ - we shall see later how the breakdown of this symmetry makes real flows so interesting. An additional symmetry is the Galilean invariance with respect to passing to a reference frame moving with the speed $V$: $\mathbf{v} \rightarrow \mathbf{v} + \mathbf{V}$ and $\mathbf{r} \rightarrow \mathbf{r} - \mathbf{V} t$. The equations of ideal hydrodynamics are also invariant with respect to re-scaling $\mathbf{r} \rightarrow \mathbf{r}/a, t \rightarrow t/b, \mathbf{v} \rightarrow \mathbf{v}/b$.

Eulerian and Lagrangian descriptions. We thus encountered two alternative types of description. The equations (1.3, 1.6) use the coordinate system fixed in space, like field theories describing electromagnetism or gravity. This type of description is called Eulerian in fluid mechanics. Another approach is called Lagrangian; it is a generalization of the approach taken in particle mechanics.
In this method one follows fluid particles and treats their current coordinates, \( \mathbf{r}(\mathbf{R}, t) \), as functions of time and their initial positions \( \mathbf{R} = \mathbf{r}(\mathbf{R}, 0) \). The substantial derivative is thus the Lagrangian derivative since it sticks to a given fluid particle, that is keeps \( \mathbf{R} \) constant: \( \frac{d}{dt} = (\partial/\partial t)_R \). Conservation laws written for a unit-mass quantity \( A \) have a Lagrangian form:

\[
\frac{dA}{dt} = \frac{\partial A}{\partial t} + (\mathbf{v} \nabla) A = 0 .
\]

Every Lagrangian conservation law together with mass conservation generates an Eulerian conservation law for a unit-volume quantity \( \rho A \):

\[
\frac{\partial (\rho A)}{\partial t} + \text{div}(\rho A \mathbf{v}) = A \left[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) \right] + \rho \left[ \frac{\partial A}{\partial t} + (\mathbf{v} \nabla) A \right] = 0 .
\]

On the contrary, if the Eulerian conservation law has the form

\[
\frac{\partial (\rho B)}{\partial t} + \text{div}(\mathbf{F}) = 0 \quad (1.14)
\]

and the flux is not equal to the density times velocity, \( \mathbf{F} \neq \rho \mathbf{v} \), then the respective Lagrangian conservation law does not exist. That means that fluid particles can exchange \( B \) conserving the total space integral – we shall see that the conservation laws of energy and momentum have that form.

### 1.2.1 Energy and momentum fluxes

Since we expect fluid particles to exchange energy and momentum then the respective fluxes must be different from "velocity times density of energy/momentum" like in (1.14). What is the difference?

The Euler equation is itself a momentum-conservation equation and must have the form of a continuity equation written for the momentum density. The momentum of the unit volume is the vector \( \rho \mathbf{v} \) whose every component is conserved so it should satisfy the equation of the form

\[
\frac{\partial \rho \mathbf{v}_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0 .
\]

Let us find the momentum flux \( \Pi_{ik} \) – the flux of the \( i \)th component of the momentum across the surface with the normal along \( k \). Substitute the mass continuity equation \( \partial \rho / \partial t = -\partial (\rho \mathbf{v}_k) / \partial x_k \) and the Euler equation \( \partial \mathbf{v}_i / \partial t = -\mathbf{v}_k \partial \mathbf{v}_i / \partial x_k - \rho^{-1} \partial p / \partial x_i \) into

\[
\frac{\partial \rho \mathbf{v}_i}{\partial t} = \rho \frac{\partial \mathbf{v}_i}{\partial t} + \mathbf{v}_i \frac{\partial \rho}{\partial t} = -\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k} \rho \mathbf{v}_i \mathbf{v}_k ,
\]
that is

$$\Pi_{ik} = \rho\delta_{ik} + \rho v_i v_k. \quad (1.15)$$

Plainly speaking, along \( v \) there is only the flux of parallel momentum \( p + \rho v^2 \) while perpendicular to \( v \) the momentum component is zero at the given point and the flux is \( p \). For example, if we direct the \( x \)-axis along the velocity at a given point then \( \Pi_{xx} = p + v^2, \Pi_{yy} = \Pi_{zz} = p \) and all the off-diagonal components are zero.

Let us now derive the equation that expresses the conservation law of energy. The energy density (per unit volume) in the flow is \( \rho(E + \frac{v^2}{2}) \). For isentropic flows, one can use \( \frac{\partial \rho E}{\partial \rho} = E + \rho \frac{\partial E}{\partial \rho} = E - \rho^{-1} \frac{\partial E}{\partial \rho} = E + P/\rho = W \) and calculate the time derivative

$$\frac{\partial}{\partial t} \left( \rho E + \frac{\rho v^2}{2} \right) = \left( W + \frac{v^2}{2} \right) \frac{\partial \rho}{\partial t} + \rho v \cdot \frac{\partial v}{\partial t} = -\text{div} \, \rho v(W + \frac{v^2}{2}).$$

Since the right-hand side is a total derivative, the integral of the energy density over the whole space is conserved. The same Eulerian conservation law in the form of a continuity equation can be obtained in a general (non-isentropic) case as well. It is straightforward to calculate the time derivative of the kinetic energy:

$$\frac{\partial}{\partial t} \left( \rho \frac{v^2}{2} \right) = -\frac{v^2}{2} \text{div} \, \rho v - v \cdot \nabla p - \rho v \cdot (v \nabla v)
= -\frac{v^2}{2} \text{div} \, \rho v - v(\rho \nabla W - \rho T \nabla s) - \rho v \cdot \nabla v^2/2.
= -\text{div} \, \rho vv^2/2 - v(\rho \nabla W - \rho T \nabla s). \quad (1.16)$$

For calculating \( \frac{\partial (\rho E)}{\partial t} \) we use \( dE = T ds - pdV = T ds + p\rho^{-2} d\rho \) so that

$$d(\rho E) = Ed\rho + \rho dE = Wd\rho + \rho T ds$$
and

$$\frac{\partial (\rho E)}{\partial t} = W \frac{\partial \rho}{\partial t} + \rho T \frac{\partial s}{\partial t} = -W \text{div} \, \rho v - \rho T v \cdot \nabla s
= -\text{div} \, \rho v W + v(\rho \nabla W - \rho T \nabla s). \quad (1.17)$$

The right sides of (1.16,refpot) contain divergences of the respective fluxes plus the exchange term (the last bracket) coming with opposite signs. Adding kinetic and potential energies together, one gets the exchange terms canceled:

$$\frac{\partial}{\partial t} \left( \rho E + \frac{\rho v^2}{2} \right) = -\text{div} \, \rho v(W + \frac{v^2}{2}). \quad (1.18)$$
As usual, the rhs is the divergence of the flux, indeed:

\[
\frac{\partial}{\partial t} \int \left( \rho E + \frac{\rho v^2}{2} \right) \, dV = - \oint \rho (W + v^2/2) \mathbf{v} \cdot d\mathbf{f}.
\]

As expected, the energy flux,

\[\rho \mathbf{v} (W + \frac{v^2}{2}) = \rho \mathbf{v} (E + \frac{v^2}{2}) + p \mathbf{v},\]

is not equal to the energy density times \( \mathbf{v} \) but contains an extra pressure term that describes the work done by pressure forces on the fluid. In other terms, any unit mass of the fluid carries an amount of energy \( W + \frac{v^2}{2} \) rather than \( E + \frac{v^2}{2} \). That means, in particular, that for energy there is no (Lagrangian) conservation law for unit mass \( \frac{d}{dt} (\cdot) = 0 \) that is valid for passively transported quantities such as entropy. This is natural because different fluid elements exchange energy by doing work.

### 1.2.2 Kinematics

We consider here the kinematics of a small fluid element. In particular, it will help us to appreciate the new conservation law, described in the next subsection. The relative motion near a point is determined by the velocity difference between neighbouring points:

\[\delta v_i = r_j \frac{\partial v_i}{\partial x_j}.\]

It is convenient to analyze the tensor of the velocity derivatives by decomposing it into symmetric and antisymmetric parts: \( \frac{\partial v_i}{\partial x_j} = S_{ij} + A_{ij} \). The symmetric tensor \( S_{ij} = (\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i})/2 \) is called strain. The vector initially parallel to the axis \( j \) turns towards the axis \( i \) with the angular speed \( \frac{\partial v_i}{\partial x_j} \), so that \( S_{ij} \) is the rate of variation of the angle between two initially mutually perpendicular small vectors along \( i \) and \( j \) axes. In other words, \( S_{ij} \) is the rate with which rectangle deforms into parallelogram. Of course, we can always transform a symmetric tensor into a diagonal form by an orthogonal transformation (i.e. by the rotation of the axes). The diagonal components are the rates of stretching in different directions. Indeed, the equation for the distance between two points along a principal direction has a form: \( \dot{r}_i = \delta v_i = r_i S_{ii} \) (no summation over \( i \)). The solution is as follows:

\[ r_i(t) = r_i(0) \exp \left[ \int_0^t S_{ii}(t') \, dt' \right]. \]
1 Basic notions and steady flows

For a permanent strain, the growth or decay is exponential in time. One recognizes that a purely straining motion converts a spherical material element into an ellipsoid, where the principal diameters grow (or decay) in time and do not rotate. Indeed, consider a circle of radius $R$ at $t = 0$. The point that starts at $x_0, y_0 = \sqrt{R^2 - x_0^2}$ goes into

$$
\begin{align*}
x(t) &= e^{S_{11}}x_0, \\
y(t) &= e^{S_{22}}y_0 = e^{S_{22}}\sqrt{R^2 - x_0^2} = e^{S_{22}}\sqrt{R^2 - x^2(t)e^{-2S_{11}t}}, \\
x^2(t)e^{-2S_{11}t} + y^2(t)e^{-2S_{22}t} &= R^2.
\end{align*}
$$

The equation (1.19) describes how the initial fluid circle turns into an ellipse whose eccentricity increases exponentially with the rate $|S_{11} - S_{22}|$ (Figure 1.5).

The sum of the strain diagonal components is $\text{div} \, \mathbf{v} = S_{ii}$ which determines the rate of the volume change:

$$Q^{-1}\frac{dQ}{dt} = -\rho^{-1}\frac{d\rho}{dt} = \text{div} \, \mathbf{v} = S_{ii}.$$  

The antisymmetric part $A_{ij} = (\partial v_j / \partial x_i - \partial v_i / \partial x_j) / 2$ has only three independent components so it could be represented via some vector $\mathbf{\omega}$: $A_{ij} = -\epsilon_{ijk}\mathbf{\omega}_k / 2$. The coefficient $-1/2$ is introduced to simplify the relation between $\mathbf{v}$ and $\mathbf{\omega}$:

$$\mathbf{\omega} = \nabla \times \mathbf{v}.$$  

The vector $\mathbf{\omega}$ is called the *vorticity* as it describes the rotation of the fluid element: $\delta \mathbf{v} = \mathbf{\omega} \times \mathbf{r}/2$. It is twice the effective local angular velocity of the fluid. A plane shearing motion, such as $v_x(y)$, corresponds to strain and vorticity being equal in magnitude (Figure 1.6).

### 1.2.3 Kelvin’s theorem

This theorem describes the conservation of velocity circulation for isentropic flows. For a rotating cylinder of a fluid, the angular momentum is proportional to
1.2 Conservation laws and potential flows

Figure 1.6 Deformation and rotation of a fluid element in a shear flow. Shearing motion is decomposed into a straining motion and rotation.

the velocity circulation around the cylinder circumference. The angular momentum and circulation are both conserved when there are only normal forces, as was already mentioned at the beginning of Section 1.1.1. Let us show that this is also true for every ‘fluid’ contour that is made of fluid particles. As fluid moves, both the velocity and the contour shape change:

$$\frac{d}{dt} \oint v \cdot dl = \oint v (dl/dt) + \oint \left( \frac{dv}{dt} \right) \cdot dl = 0.$$  \hspace{1cm} (1.20)

The first term here disappears because it is a contour integral of the complete differential: since $dl/dt = \delta v$ then $\oint v (dl/dt) = \oint \delta (v^2/2) = 0$. In the second term we substitute the Euler equation for isentropic motion, $dv/dt = -\nabla W$, and use Stokes’ formula, which tells that the circulation of a vector around a closed contour is equal to the flux of the curl through any surface bounded by the contour: $\oint \nabla W \cdot dl = \int \nabla \times \nabla W \, df = 0$.

Stokes’ formula also tells us that $\oint v dl = \int \omega \cdot df$. Therefore, the conservation of the velocity circulation equals the conservation of the vorticity flux. To better appreciate this, consider an alternative derivation. Taking the curl of (1.11) we get

$$\frac{\partial \omega}{\partial t} = \nabla \times (v \times \omega). \hspace{1cm} (1.20)$$

The simplest lesson one can immediately draw from (1.20) is that if $\omega \equiv 0$ then $\partial \omega/\partial t \equiv 0$ that is irrotational flow remains such in an ideal fluid. Not only zero but any value of vorticity is conserved for fluid particles. Indeed, (1.20) is the same equation that describes the magnetic field in a perfect conductor: substituting the condition for the absence of the electric field in the frame moving with the velocity $v$, $cE + v \times H = 0$, into the Maxwell equation $\partial H/\partial t = -c \nabla \times E$, one gets $\partial H/\partial t = \nabla \times (v \times H)$. The magnetic flux is conserved in a perfect conductor and so is the vorticity flux in an isentropic flow. One can visualize the vector field by introducing field lines, which give the direction of the field at any point while their density is proportional to the magnitude of the
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field. Kelvin’s theorem means that vortex lines move with material elements in an inviscid fluid exactly like magnetic lines are frozen into a perfect conductor. One way to prove this is to show that \( \omega/\rho \) (and \( H/\rho \)) satisfies the same equation as the distance \( r \) between two fluid particles: \( dr/dt = (r \cdot \nabla)v \). This is done using \( d\rho/dt = -\rho \text{div} v \) and applying the general relation

\[
\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B \tag{1.21}
\]

to \( \nabla \times (v \times \omega) = (\omega \cdot \nabla)v - (v \cdot \nabla)\omega - \omega \text{div} v \). We then obtain

\[
\frac{d}{dt} \frac{\omega}{\rho} = \frac{1}{\rho} \frac{d\omega}{dt} - \frac{\omega}{\rho^2} \frac{d\rho}{dt} = \frac{1}{\rho} \left[ \frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega \right] + \frac{\omega \text{div} v}{\rho} = \frac{1}{\rho} \left[ (\omega \cdot \nabla)v - (v \cdot \nabla)\omega - \omega \text{div} v + (v \cdot \nabla)\omega \right] + \frac{\omega \text{div} v}{\rho} = \left( \frac{\omega}{\rho} \cdot \nabla \right)v. \tag{1.22}
\]

Since \( r \) and \( \omega/\rho \) move together, then any two close fluid particles chosen on the vorticity line always stay on it. Consequently any fluid particle stays on the same vorticity line so that any fluid contour never crosses vorticity lines and the flux is indeed conserved. Compression transversal to the lines decreases the contour area thus increasing vorticity, similar effect leads to magnetohydrodynamic dynamo.

It is important to stress that Kelvin theorem is a nonlocal conservation law so it is not equivalent to the conservation of an angular momentum, which has a local density \( \rho v \times r \). The symmetry which corresponds to the conservation of vorticity flux corresponds to re-labelling lagrangian coordinates.

We have finished the formulations of the equations and their general properties and will now consider the simplest case that allows for an analytic study. This involves several assumptions.

1.2.4 Irrotational and incompressible flows

Irrotational flows are defined as having zero vorticity: \( \omega = \nabla \times v \equiv 0 \). In such flows, \( \oint v \cdot dl = 0 \) round any closed contour, which means, in particular, that there are no closed streamlines for a singly connected domain. Note that the flow has to be isentropic to stay irrotational (i.e. inhomogeneous heating can generate vortices). A zero-curl vector field is potential, \( v = \nabla \phi \), so that the Euler
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Equation (1.11) takes the form

\[ \nabla \left( \frac{\partial \phi}{\partial t} + \frac{v^2}{2} + W \right) = 0. \]

After integration, one gets

\[ \frac{\partial \phi}{\partial t} + \frac{v^2}{2} + W = C(t) \]

and the space-independent function \( C(t) \) can be included into the potential, \( \phi(r, t) \rightarrow \phi(r, t) + \int C(t') dt' \), without changing velocity. Eventually,

\[ \frac{\partial \phi}{\partial t} + \frac{v^2}{2} + W = 0. \quad (1.23) \]

For a steady flow, we thus obtained a more strong Bernoulli theorem with \( v^2/2 + W \) being the same constant along all the streamlines, as distinct from a general case, where it may be a different constant along different streamlines.

Absence of vorticity provides for a dramatic simplification, which we exploit in this section and the next one. Unfortunately for pipeline operators and fortunately for birds, irrotational flows are much less frequent than Kelvin’s theorem suggests. The main reason is that (even for isentropic flows) the viscous boundary layers near solid boundaries generate vorticity, as we shall see in Section 1.5. Yet we shall also see there that large regions of the flow can be unaffected by the vorticity generation and effectively described as irrotational.

Another class of potential flows is provided by small-amplitude oscillations (like waves or motions due to oscillations of an immersed body). If the amplitude of oscillations \( a \) is small compared with the velocity scale of change \( l \) then \( \partial v / \partial t \approx v^2 / a \) while \( (v \nabla) v \approx v^2 / l \) so that the non-linear term can be neglected and \( \partial v / \partial t = -\nabla W \). Taking the curl of this equation we see that \( \omega \) is conserved but its average is zero in oscillating motion so that \( \omega = 0 \).

After we simplified the Euler equation as much as possible, from (1.2) to (1.23), let us simplify the continuity equation.

**Incompressible fluid** can be considered as such if the density of any fluid element does not change in a flow: \( d \ln \rho / dt = -\nabla v = 0 \). For an incompressible fluid, the continuity equation is thus reduced to

\[ \nabla \cdot \vec{v} = 0. \quad (1.24) \]

Strictly speaking, this could be true even when density varies in space and in time as long as any element keeps its density as it moves. In other words, \( \partial \rho / \partial t \) and
Basic notions and steady flows

\[ (v \nabla) \rho \] can both be nonzero as long as their sum is zero. We, however, consider below only the simplest case when density is constant both in time and in space. This means that in the continuity equation, \( \partial \rho / \partial t + (v \nabla) \rho + \rho \text{div} \ v = 0 \), the first two terms are much smaller than the third one. Let the velocity \( v \) change over the scale \( l \) and the time \( \tau \). The density variation can be estimated as

\[ \delta \rho \simeq (\partial \rho / \partial \rho)_{s} \delta \rho \simeq (\partial \rho / \partial p)_{s} \rho v^2 \simeq \rho v^2 / c^2, \quad (1.25) \]

where the pressure change was estimated from the Bernoulli relation. Requiring \( (v \nabla) \rho \simeq v \delta \rho / l \ll \rho \text{div} v \simeq \rho v / l \), we get the condition \( \delta \rho \ll \rho \) which, according to (1.25), is true as long as the velocity is much less than the speed of sound:

For time-dependent flows, one must also require that the density changes slowly enough: \( \partial \rho / \partial t \ll \rho \text{div} v \). Comparing \( \partial \rho / \partial t \simeq v / \tau \) and \( \nabla p / \rho \simeq c^2 \delta \rho / \rho l \), we estimate the density change due to temporal change as \( \delta \rho \simeq l \rho v / \tau c^2 \), so that

\[ \partial \rho / \partial t \simeq \delta \rho / \tau \simeq l \rho v / \tau c^2 \ll \rho \text{div} v \simeq \rho v / l. \]

Therefore, the second condition condition of incompressibility is that the typical time of change \( \tau \) must be much larger than the typical scale of change \( l \) divided by the sound velocity \( c \):

\[ \tau \gg l / c, \quad (1.26) \]

Indeed, sound equilibrates densities in different points so that all flow changes must be slow to let sound pass.

For isentropic motion of an incompressible fluid, the internal energy does not change (\( dE = T \, ds + p \rho^{-2} d \rho \)) so that one can put everywhere \( W = p / \rho \).

Since density is no longer an independent variable, the equations that contain only velocity can be chosen: one takes (1.20) and (1.24).

In two dimensions, incompressible flow can be characterized by a single scalar function. Since \( \partial v_x / \partial x = - \partial v_y / \partial y \) then we can introduce the stream function \( \psi \) defined by \( v_x = \partial \psi / \partial y \) and \( v_y = - \partial \psi / \partial x \). Recall that the streamlines are defined by \( v_x dy - v_y dx = 0 \), which now corresponds to \( d \psi = 0 \), that is the equation \( \psi(x, y) = \text{const.} \) does indeed determine streamlines. Another important use of the stream function is that the flux through any line is equal to the difference of \( \psi \) at the endpoints (and is thus independent of the line form –
an evident consequence of incompressibility):

$$\int^2_1 v_n dl = \int^2_1 (v_x dy - v_y dx) = \int d\psi = \psi_2 - \psi_1. \quad (1.27)$$

Here $v_n$ is the velocity projection on the normal; that is the flux is equal to the modulus of the vector product $\int |v \times dl|$, see Figure 1.7. A solid boundary at rest has to coincide with one of the streamlines.

**Potential flow of an incompressible fluid.** By virtue of (1.24) the potential satisfies the Laplace equation

$$\Delta \phi = 0,$$

with the condition $\partial \phi / \partial n = 0$ on a solid boundary at rest.

It is the distinctive property of an irrotational incompressible flow that the velocity distribution is defined completely by a linear equation. Owing to linearity, velocity potentials can be superimposed (but not pressure distributions). Before considering particular flows, we formulate several general statements. First, the Laplace equation is elliptic, which means that the solutions are smooth inside the domains, singularities could exist on boundaries only, in contrast to hyperbolic (say, wave) equations. Remind that the second-order linear differential operator $\sum a_i \partial^2$ is called elliptic if all $a_i$ are of the same sign, hyperbolic if their signs are different and parabolic if at least one coefficient is zero. The names come from the fact that a real quadratic curve $ax^2 + 2bxy + cy^2 = 0$ is a hyperbola, an ellipse or a parabola depending on whether $ac - b^2$ is negative, positive or zero. For hyperbolic equations, one can introduce characteristics where solution stays constant; if different characteristics cross then a singularity may appear inside the domain. Solutions of elliptic equations are smooth, their stationary points are saddles rather than maxima or minima. See also Sects. ?? and ??.
Second, integrating (1.29) over any volume one gets

\[ \int \Delta \phi \, dV = \int \text{div} \nabla \phi \, dV = \oint \nabla \phi \cdot df = 0, \]

that is the flux is zero through any closed surface (as is expected for an incompressible fluid). That means, in particular, that \( v = \nabla \phi \) changes sign on any closed surface so that extrema of \( \phi \) could be on the boundary only. The same can be shown for velocity components (e.g. for \( \partial \phi / \partial x \)) since they also satisfy the Laplace equation. That means that for any point \( P \) inside one can find \( P' \) having higher \( |v_x| \). If we choose the \( x \)-direction to coincide with \( \nabla \phi \) at \( P \) we conclude that for any point inside one can find another point in the immediate neighbourhood where \( |v| \) is greater. In other terms, \( v^2 \) cannot have a maximum inside (but can have a minimum). Similarly for pressure, taking the Laplacian of the Bernoulli relation (1.30),

\[ \Delta p = -\rho \Delta v^2 / 2 = -\rho (\nabla v)^2, \]

and integrating it over volume, one obtains

\[ \oint \nabla p \cdot df = -\rho \int (\nabla v)^2 \, dV < 0, \]

that is a pressure minimum could be only on a boundary (although a maximum can occur at an interior point). For steady flows, \( v^2 / 2 + p / \rho = \text{const.} \) so that the points of max \( v^2 \) coincide with those of min \( p \) and all are on a boundary.\(^7\) The knowledge of points of minimal pressure is important for cavitation, which is a creation of gas bubbles when the pressure falls below the vapour pressure; when such bubbles then experience higher pressure, they may collapse, producing shock waves that cause severe damage to moving boundaries like turbine blades and ships' propellers. Shock are also created when the local fluid velocity exceeds the velocity of sound, as we shall see in Section ?? below; this must happen first at the velocity maxima, which again are possible only on the boundary of a potential flow.

Despite the high degree of idealization, the theory of incompressible potential flows is of significant practical importance. Not only it describes large regions of the flows outside wakes past the bodies, as described in Section 1.5.4, but also high-power explosions as described in Exercise 1.18.

Two-dimensional case Particularly beautiful is the description of two-dimensional (2D) potential incompressible flows. Both potential and stream
function exist in this case. The equations

\[ v_x = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v_y = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}, \]

(1.28)
could be recognized as the Cauchy–Riemann conditions for the complex potential \( w = \phi + i\psi \) to be an analytic function of the complex argument \( z = x + iy \). That means that the rate of change of \( w \) does not depend on the direction in the \( x, y \)-plane, so that one can define the complex derivative \( dw/dz \), which exists everywhere. For example, both choices \( dz = dx \) and \( dz = idy \) give the same answer by virtue of (1.28):

\[ \frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial y} = v_x - iv_y = ve^{-i\theta}, \quad \mathbf{v} = v_x + iv_y = \frac{dw}{dz}. \]

The complex form allows one to describe many flows in a compact form and find flows in a complex geometry by mapping a domain onto a standard one. Such a transformation must be conformal, i.e., done using an analytic function so that the equations (1.28) preserve their form in the new coordinates. Conformal transformations stretch uniformly in all directions at every point but the magnitude of stretching generally depends on a point. As a result, conformal maps preserve angles but not the distances. These properties had been first made useful in naval cartography (Mercator 1569) well before the invention of the complex analysis. Indeed, to discover a new continent it is preferable to know the direction rather than the distance ahead.

We thus get our first (infinite) family of flows: any complex function analytic in a domain and having a constant imaginary part on the boundary describes a potential flow of an incompressible fluid in this domain. Uniform flow is just \( w = (v_x - iv_y)z \). Here are two other examples:

1. potential flow near a stagnation point \( \mathbf{v} = 0 \) (inside the domain or on a smooth boundary) is expressed via the rate-of-strain tensor \( S_{ij} \): \( \phi = S_{ij}x_ix_j/2 \) with \( \text{div} \mathbf{v} = S_{ii} = 0 \). In the principal axes of the tensor, one has \( v_x = kx, \ v_y = -ky \), which corresponds to \( \phi = k(x^2 - y^2)/2, \ \psi = kxy, \ w = kz^2/2 \).
The streamlines $x = \psi / ky$ and trajectories $x(t) = x(0)y(0)/y(t)$ are rectangular hyperbolae. This is applied, in particular, on the boundary, which has to coincide with one of the principal axes ($x$ or $y$) or both. The figure presents the flows near the boundary along $x$ and along $x$ and $y$ (half of the previous one):

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.8}
\caption{Flows described by the complex potential $w = A\zeta^n$.}
\end{figure}

(2) Consider the potential in the form $w = A\zeta^n$, that is $\phi = Ar^n \cos n\theta$ and $\psi = Ar^n \sin n\theta$. Zero-flux boundaries should coincide with the streamlines so two straight lines, $\theta = 0$ and $\theta = \pi/n$ could be seen as boundaries. Choosing different $n$, one can have different interesting particular cases. The velocity modulus $v = \left| \frac{dw}{dz} \right| = n|A|r^{n-1}$ at $r \to 0$ either turns to 0 ($n > 1$) or to $\infty$ ($n < 1$) (Figure 1.8).

One can think of those solutions as obtained by a conformal transformation $\zeta = z^n$, which maps the $z$-domain into the full $\zeta$-plane. The potential $w = A\zeta^n = Az$ describes a uniform flow in the $\zeta$-plane. Respective $z$ and $\zeta$ points have the same value of the potential so that the transformation maps streamlines onto streamlines. The velocity in the transformed domain is $\frac{dw}{d\zeta} = (\frac{dw}{dz})(\frac{dz}{d\zeta})$, that is the velocity modulus is inversely proportional to the stretching factor of the transformation. This has two important consequences: first, the energy of the potential flow is invariant with respect to conformal transformations, i.e. the energy inside every closed curve in the $z$-plane is the
1.3 Moving through fluids

How much force one needs to set a body in motion through a fluid? Consider an air bubble in champaign. The mass of the air in the bubble is thousand times less than the mass of the displaced liquid which determines the Archimedes force. Will the bubbles then speed up with the acceleration of 1000g, literally blowing the liquid into our face? Indeed, moving body must involve certain amount of moving fluid. We then are tempted to conclude that the force acting on the body must be equal to the time derivative of the momentum of the body plus the momentum of the fluid. Reflecting a bit more, we see that this is incorrect too: if, for instance, the liquid is enclosed by rigid walls then its momentum is identically zero! But if those walls are far away from the body, they must not influence the force. We then expect that the body sets in motion some fluid in the vicinity (and this determines the force), while the compensating reflux is spread over the whole fluid and does not influence the force. In this section, we
describe the flows of ideal incompressible fluid set in motion by moving bodies. We start from the most symmetric case of a moving sphere and then consider a body of arbitrary shape. From each flow, we shall derive the contribution of the fluid into the force needed to set the body in motion. That contribution is equivalent to an appearance of added mass, apparently the first example of renormalization in physics. We shall find out that it is the quasi-momentum (not momentum) of the fluid that determined the added mass. We discuss the difference between the conservation of momentum (due to space homogeneity) and quasi-momentum (due to fluid homogeneity).

For non-symmetric moving bodies, fluid generally applies force not only opposite to the direction of motion but also across. The force perpendicular to the motion is called lift, since it keeps birds and planes from falling from the skies. In this section, we find out that there is neither lift nor drag for a steady motion in an ideal fluid which will induce us to introduce friction in the next section.
1.3 Moving through fluids

1.3.1 Incompressible potential flow past a body

If we assume that the flow appears from rest, Kelvin theorem suggests that the vorticity must be identically zero. Alternatively, in the reference frame of the body, fluid comes from infinity, where a uniform flow also has zero vorticity. Flow is thus assumed to be four ‘i’: infinite, irrotational, incompressible and ideal. The algorithm to describe such a flow is to solve the Laplace equation

$$\Delta \phi = 0.$$

(1.29)

The boundary condition on the body surface is the requirement that the normal components of the body and fluid velocities coincide, that is at any given moment one has $\partial \phi / \partial n = u_n$, where $u$ is the body velocity. After finding the potential, one calculates $v = \nabla \phi$ and then finds pressure from the Bernoulli equation:

$$p = -\rho (\partial \phi / \partial t + v^2 / 2).$$

(1.30)

1.3.2 Moving sphere

Solutions of the equation $\Delta \phi = 0$ that vanish at infinity are $1/r$ and its derivatives, $\partial^n (1/r) / \partial x^n$ in three dimensions. Owing to the complete symmetry of the sphere, its motion is characterized by a single vector of its velocity $u$. Linearity requires $\phi \propto u$ so the flow potential could be only made as a scalar product of the vectors $u$ and the gradient, which is the dipole field:

$$\phi = a \frac{u \cdot \nabla}{r} = -a \frac{(u \cdot n)}{r^2}$$

where $n = r / r$. On the body, $r = R$ and $v \cdot n = u \cdot n = u \cos \theta$. Using $\phi = -ua \cos \theta / r^2$ and $v_R = 2auR^{-3} \cos \theta$, this condition gives $a = R^3/2$.

Now one can calculate the pressure

$$p = p_0 - \rho v^2 / 2 - \rho \partial \phi / \partial t,$$
having in mind that our solution moves with the sphere that is \( \phi(t - u, u) \) and

\[
\frac{\partial \phi}{\partial t} = \dot{u} \cdot \frac{\partial \phi}{\partial u} - u \cdot \nabla \phi,
\]

which gives

\[
p = p_0 + \rho u^2 \left( \frac{9}{8} \cos^2 \theta - \frac{5}{8} \right) + \frac{\rho R}{2} n \cdot \dot{u}.
\]

The force acting on the body is minus the pressure integral over the surface \( -\oint p \, df \). For example, assuming that the body velocity does not change its direction, \( n \cdot \dot{u} = \dot{u} \cos \theta \), we derive

\[
F_x = -\oint p \cos \theta \, df = -\rho R^3 \dot{u} \pi \int \cos^2 \theta \, d \cos \theta = -2\pi \rho R^3 \dot{u} / 3. \tag{1.31}
\]

We thus see that the mass of the fluid one needs to accelerate is half the mass of the displaced fluid. This is one of the simplest examples of renormalization in physics: the body moving through a fluid acquires additional (also called induced) mass. A spherical air bubble in a liquid has a mass that is half of the mass of the displaced liquid; since the buoyancy force is the displaced mass times \( g \), then we conclude with a relief that the initial bubble acceleration is close to \( 2g \). As the bubble accelerates relative to the liquid, the drag force increases, as will be described in Section 1.5 below, and the accelerates decreases.

If the radius depends on time too then \( F_x \propto \partial \phi / \partial t \propto -\partial (R^3 u) / \partial t \). Remarkably, it means that a shrinking moving body experiences an accelerating force, this will be discussed after (1.44).

According to our formulas, for a uniformly moving sphere with a constant radius, \( \dot{R} = \dot{u} = 0 \), the force is zero: \( \oint p \, df = 0 \). In a two-dimensional case (say, flow around a cylinder with the radius \( R \)), the potential is \( \phi = -R^2 (u \cdot \nabla) \log r \) and the pressure on the surface is \( p = -2\rho \sin^2 \theta + \rho R n \cdot \dot{u} \). Similar to (1.33), after angular integration only the last term contributes the total force which thus vanishes for a steady motion. The force is zero because of fore-and-aft symmetry: incoming and outgoing parts of the steady flow are identical and provide for the same pressure fields. This flies in the face of our common experience: fluids do resist even bodies moving with a constant speed. Maybe we obtained zero force in a steady case due to a symmetrical shape of the body?

### 1.3.3 Moving body of an arbitrary shape

In two dimensions, potential flow around a body with an arbitrary shape can be obtained by a conformal map of the solution for a circle. If a steady motion of a
circle meets no force, the same is true for any body shape. Indeed, for a steady flow, the pressure (up to a constant) is \( p = -\rho v^2/2 = -\rho |d\omega/dz|^2/2 \). One can combine the forces into the integral over the body surface where \( d\omega = d\tilde{\omega} \) (Blasius 1910):

\[
F_x - iF_y = -i \oint p d\tilde{z} = \frac{i\rho}{2} \oint \frac{d\omega}{dz} d\tilde{\omega} = \frac{i\rho}{2} \oint \left( \frac{d\omega}{dz} \right)^2 dz.
\]

Since we integrate an analytic function having no singularities we can enlarge the contour to any extent, so that the integral vanishes, since velocity decreases faster than \( 1/|z| \) due to mass conservation. Note that this consideration does not rule out nonzero torque acting on a body.

In three dimensions, flow past a body of an arbitrary shape generally cannot be found analytically. However, the main beauty of the potential theory (and conformal analysis used in the previous paragraph) is that one can say something about ‘here’ by considering the field ‘there’. In our case, we are interested in the forces acting on the body surface yet we consider the flow far away, which must be weakly dependent on the body shape. Indeed, at large distances from the body, a solution of \( \Delta \phi = 0 \) is again sought in the form of the first non-vanishing multipole. The first (charge) term \( \phi = a/r \) cannot be present because it corresponds to the velocity \( v = -ar/r^3 \) with the radial component \( v_r = a/r^2 \) providing for an \( r \)-independent flux \( 4\pi \rho a \) through a closed sphere of radius \( r \); existence of a flux contradicts mass conservation. So the first non-vanishing term is again a dipole:

\[
\phi = A \cdot \nabla (1/r) = -(A \cdot n)r^{-2},
\]

\[
v = 3(A \cdot n)n - Ar^{-3}.
\]

For the sphere above, \( A = uR_0^3/2 \), where \( R_0 \) is the radius. For non-symmetric bodies, the vectors \( A \) and \( u \) are not collinear, though linearly related \( A_i = a_{ik} u_k \), where the tensor \( a_{ik} \) (having the dimensionality of volume) depends on the body shape.

To relate the force acting on the body to the flow at large distances, let us start by calculating the energy \( E = \rho \int v^2 dV/2 \) of the moving fluid outside the body and inside the large sphere of radius \( R \). We present \( v^2 = u^2 + (v - u)(v + u) \) and
write \( v + u = \nabla(\phi + u \cdot r) \). Using \( \text{div} \, v = \text{div} \, u = 0 \) one can write

\[
\int_{r < R} u^2 \, dV = u^2 (V - V_0) + \int_{r < R} \text{div}(\phi + u \cdot r)(v - u) \, dV
\]

\[
= u^2 (V - V_0) + \oint_{S + S_0} (\phi + u \cdot r)(v - u) \, df
\]

Substituting

\[
\phi = - (\mathbf{A} \cdot \mathbf{n}) R^{-2}, \quad v = 3 n (\mathbf{A} \cdot \mathbf{n}) - \mathbf{A} R^{-3}
\]

and integrating over angles,

\[
\int (\mathbf{A} \cdot \mathbf{n})(u \cdot n) \, d\Omega = A_i u_k \int n_i n_k \, d\Omega = A_i u_k \delta_{ik} \int \cos^2 \theta \sin \theta \, d\theta d\varphi
\]

we obtain the energy in the form

\[
E = \rho 4\pi (\mathbf{A} \cdot \mathbf{u}) - V_0 u^2 / 2 = m_{ik} u_i u_k / 2. \tag{1.32}
\]

Here we introduce the induced-mass tensor:

\[
m_{ik} = 4\pi \rho a_{ik} - \rho V_0 \delta_{ik}.
\]

For a sphere, \( m_{ik} = \rho V_0 \delta_{ik} / 2 \), that is half the mass of the displaced fluid. Induced mass can be much larger (for a thin disc moving perpendicular to its plane) and much smaller (for a needle moving "end on") than the displaced mass.

\[
V = 4\pi R^3 / 3, \quad E = \rho [2\pi (\mathbf{A} \cdot \mathbf{u}) - V_0 u^2 / 2]
\]
1.3 Moving through fluids

We now have to pass from the energy to the force acting on the body which is done by considering the change in the energy of the body (the same as minus the change of the fluid energy $dE$) being equal to the work done by force $F$ on the path $udr$: $dE = -F \cdot u \, dr$. The change of the momentum of the body is $dP = -F dt$ so that $dE = u \cdot dP$. This relation is true for changes caused by the velocity change by force (not by the change in the body shape) so that the change of the body momentum is $dP_i = m_{ik} u_k$ and the force is

$$F_i = -m_{ik} u_k. \quad (1.33)$$

We see that the presence of potential flow means only an additional mass but not resistance for an arbitrary body shape. The reason is that a steady potential flow in an incompressible fluid has fore-and-aft symmetry at infinity so that the total momentum does not change inside the volume containing the body.

How to generalize (1.33) for the case when both $m_{ik}$ and $u$ change? Our consideration for a sphere suggests that, since the respective contribution into the pressure is $-\rho \partial \phi/\partial t \propto \partial (m_{ik} u_k)/\partial t$, then the proper generalization is

$$F_i = -\frac{d}{dt} m_{ik} u_k. \quad (1.34)$$

It looks as if $m_{ik} u_k$ is the momentum of the fluid yet it is not (it is quasi-momentum as explained in the next section).

The equation of motion for the body under the action of an external force $f$,

$$\frac{d}{dt} Mu_i = f_i + F_i = f_i - \frac{d}{dt} m_{ik} u_k,$$

could be written in a form that makes the term induced mass clear:

$$\frac{d}{dt} (M \delta_{ik} + m_{ik}) u_k = f_i. \quad (1.35)$$

Remark on the case of time-dependent body mass $M(t)$ is in order. Was it correct to put the body mass inside the time derivative? The answer depends on whether the body mass grows due to condensation (as in the problem 1.14) or decreases due to evaporation or dissolution (as in the problem 2.5). In the former case, if the vapour was at rest before condensation, then it adds no momentum after condensation. Therefore, the momentum change is only due to the external force, so that the time derivative of the momentum is equal to the force acting on the body, as in (1.35). In the latter case, the material leaves with the same velocity, that is evaporation by itself does not change the body velocity. The
velocity change is due to the force, so that the velocity time derivative must be equal to the force divided by the mass, and the equation takes the form:

$$M \frac{d}{dt} u_i + \frac{d}{dt} m_{ik} u_k = f_i.$$  \hfill (1.36)

**Body in a flow.** Consider now an opposite situation when the fluid moves in an oscillating way while a small body is immersed in the fluid. For example, a long sound wave propagates in a fluid. We do not consider here the external forces that move the fluid; we wish to relate the body velocity $u$ to the fluid velocity $v$, which is supposed to be homogeneous on the scale of the body size. If the body moved with the same velocity, $u = v$, then it would be under the action of a force that would act on the fluid in its place, $\rho V_0 \dot{v}$. Relative motion gives the reaction force $d m_{ik} (v_k - u_k) / dt$. The sum of the forces gives the time derivative of the body momentum:

$$\frac{d}{dt} M u_i = \rho V_0 \dot{v}_i + \frac{d}{dt} m_{ik} (v_k - u_k).$$  \hfill (1.37)

Integrating over time with the integration constant zero (since $u = 0$ when $v = 0$) we get the relation between the velocities of the body and of the fluid:

$$(M \delta_{ik} + m_{ik}) u_k = (m_{ik} + \rho V_0 \delta_{ik}) v_k.$$

For a sphere, $u = 3 \rho / (\rho + 2 \rho_0)$, where $\rho_0$ is the density of the body. For a liquid droplet in air, $u = 3 \nu \rho / 2 \rho_0 \ll v$. For an air bubble in a liquid, $\rho_0 \ll \rho$ and $u \approx 3 \nu$. The motion of denser/lighter bodies is retarded/advanced relative to the fluid.

### 1.3.4 Quasi-momentum and induced mass

In the previous section, we obtained the force acting on an accelerating body via the energy of the fluid and the momentum of the body because the momentum of the fluid, $M = \rho \int v \, dV$, is not well-defined for a potential flow around the body. For example, the integral of $v_i$, $v_i = D(3 \cos^2 \theta - 1) r^{-1}$ depends on the form of the volume chosen: it is zero for a spherical volume and non-zero for a cylinder of length $L$ and radius $R$ set around the body:

$$\int_{-1}^{1} (3 \cos^2 \theta - 1) \, d \cos \theta = 0,$$

$$M_x = 4 \pi \rho D \int_{-L}^{L} dz \int_{0}^{R} r \, dr \frac{2z^2 - r^2}{(z^2 + r^2)^{3/2}} = -\frac{4 \pi \rho D L}{(L^2 + R^2)^{1/2}}.$$  \hfill (1.38)
Does that mean that the momentum stored in the fluid depends on the boundary conditions at infinity? In fact, it does. For example, the motion by a sphere in a fluid enclosed by rigid walls must be accompanied by the displacement of an equal amount of fluid in the opposite direction; then the momentum of the fluid must be \( -\rho V_0 u = -4\pi \rho R^3 u / 3 \) rather than \( \rho u V_0 / 2 \). The negative momentum \(-3\rho u V_0 / 2\) delivered by the walls is absorbed by the whole body of fluid and results in an infinitesimal back-flow, while the momentum \( \rho u V_0 / 2 \) delivered by the sphere results in a finite localized flow. From (1.38) we can get a shape-independent answer \( 4\pi \rho D \) only in the limit \( L / R \to \infty \). To recover the answer \( 4\pi \rho D / 3 \) (=\( \rho u V_0 / 2 = \rho u 2\pi R^3 / 3 \) for a sphere) that we expect from (1.34), we need to subtract the reflux \( 8\pi \rho D / 3 = \rho u 4\pi R^3 / 3 \), compensating the body motion.\(^9\)

It is the quasi-momentum of the fluid particles that is independent of the remote boundary conditions and whose time derivative gives the inertial force (1.34) acting on the body. Conservation laws of the momentum and the quasi-momentum follow from different symmetries. The momentum expresses invariance of the Hamiltonian \( \mathcal{H} \) with respect to the shift of coordinate system. If the space is filled by a medium (fluid or solid), then the quasi-momentum expresses invariance of the Hamiltonian with respect to a space shift, keeping the medium fixed. That invariance follows from the identity of different elements of the medium. In a crystal, such shifts are allowed only by the lattice spacing. In a continuous medium, shifts are arbitrary. In this case, the system Hamiltonian must be independent of the coordinates:

\[
\frac{\partial \mathcal{H}}{\partial x_i} = \frac{\partial \mathcal{H}}{\partial \pi_j} \frac{\partial \pi_j}{\partial x_i} + \frac{\partial \mathcal{H}}{\partial q_j} \frac{\partial q_j}{\partial x_i} = 0, \tag{1.39}
\]

where the vectors \( \pi(x, t), q(x, t) \) are respectively canonical momentum and coordinates (in every point in space). We need to define the quasi-momentum \( K \) whose conservation is due to invariance of the Hamiltonian: \( \partial K_i / \partial t = \partial \mathcal{H} / \partial x_i = 0 \). Recall that the time derivative of any function of canonical
variables is given by the Poisson bracket of this function with the Hamiltonian:

\[
\frac{\partial K_i}{\partial t} = [K_i, H] = \frac{\partial K_i}{\partial q_j} \frac{\partial H}{\partial \pi_j} - \frac{\partial K_i}{\partial \pi_j} \frac{\partial H}{\partial q_j} = \frac{\partial H}{\partial x_i} + \frac{\partial H}{\partial q_j} \frac{\partial q_j}{\partial x_i}.
\]

This gives the partial differential equations for the quasi-momentum,

\[
\frac{\partial K_i}{\partial \pi_j} = -\frac{\partial q_j}{\partial x_i}, \quad \frac{\partial K_i}{\partial q_j} = \frac{\partial \pi_j}{\partial x_i},
\]

whose solution is as follows:

\[
K_i = -\int d\pi_j \frac{\partial q_j}{\partial x_i}.
\] (1.40)

For isentropic (generally compressible) flow of an ideal fluid, the Hamiltonian description can be given in Lagrangian coordinates, which describe the current position of a fluid element (particle) \( r(R, t) \) as a function of its initial position \( R \) and time \( t \). Since we want our variable to have a finite change for a localized flow, we choose the canonical coordinate as the displacement \( q = r - R \), which is the continuum limit of the variable that describes lattice vibrations in the solid state physics. The canonical momentum is \( \pi(R, t) = \rho_0(R)v(R, t) \) where the velocity is \( v = (\partial r/\partial t) \equiv \dot{r} \). Here \( \rho_0 \) is the density in the reference (initial) state, which can always be chosen to be uniform. As we discussed in Section 1.1.2 deriving the Euler equation, the fluid particle always has the same (unit) mass.

The Hamiltonian is as follows:

\[
H = \int \rho_0[ W(q) + v^2/2] \, dR.
\] (1.41)

where \( W = E + p/\rho \) is the enthalpy, which thus plays a role of the potential energy of a fluid particle. The Hamiltonian for a fluid particle, \( W + v^2/2 \), is generally time-dependent.\(^{10}\) The density is as follows: \( \rho(R, t) = \rho_0\det(\partial r/\partial R) \).

Canonical equations of motion, \( \dot{q}_i = \partial H/\partial \pi_i \) and \( \dot{\pi}_i = -\partial H/\partial q_i \), give, respectively, \( \dot{r}_i = v_i \) and \( \dot{v}_i = -\partial W/\partial r_i = -\rho^{-1}\partial p/\partial r_i \). The velocity \( v \) is now an independent variable and not a function of the coordinates \( r \). All the time derivatives are for fixed \( R \), i.e. they are substantial derivatives. The quasi-momentum (1.40) is as follows:

\[
K_i = -\rho_0 \int v_j \frac{\partial q_j}{\partial R_i} \, dR = \rho_0 \int v_j \left( \delta_{ij} - \frac{\partial r_j}{\partial R_i} \right) \, dR,
\] (1.42)
In plain words, only those particles contribute quasi-momentum whose motion is disturbed by the body, so that for them \( \partial r_j / \partial R_i \neq \delta_{ij} \). The integral (1.42) converges for spatially localized flows since \( \partial r_j / \partial R_i \to \delta_{ij} \) when \( R \to \infty \).

Unlike (1.38), the quasi-momentum (1.42) is independent of the form of distant boundaries. Using \( \rho_0 dR = \rho dr \) one can also present

\[
K_i = \rho_0 \int v_j \left( \delta_{ij} - \frac{\partial r_j}{\partial R_i} \right) dR
= \int \rho v_i dR - \rho_0 \int v_j \frac{\partial r_j}{\partial R_i} dR,
\]

(1.43)
i.e. indeed the quasi-momentum is the momentum minus what can be interpreted as a reflux.

The conservation can now be established, substituting the equation of motion \( \rho \dot{v} = -\partial p / \partial r \) into

\[
\dot{K}_i = -\rho_0 \int \left( \dot{v}_j \left( \frac{\partial q_j}{\partial R_i} + v_j \frac{\partial v_j}{\partial R_i} \right) + v_j \frac{\partial v_j}{\partial R_i} \right) dR
= -\rho_0 \int \left[ \dot{v}_j \left( \frac{\partial r_j}{\partial R_i} - \delta_{ij} \right) + v_j \frac{\partial v_j}{\partial R_i} \right] dR
\]
1 Basic notions and steady flows

\[ = -\rho_0 \int \left( \delta_{ij} \frac{\partial p}{\partial r_j} \frac{\partial W}{\partial r_j} \frac{\partial v}{\partial R_i} + \frac{\partial v^2}{2\partial R_i} \right) dR \]
1.3 Moving through fluids

\[ \begin{align*}
= - \int \frac{\partial p}{\partial r_i} \, dr + \int \frac{\partial}{\partial r_i} \left( W - \frac{\rho \nu^2}{2} \right) \, dR \\
= - \int \frac{\partial p}{\partial r_i} \, dr = \oint p \, df_i.
\end{align*} \] (1.44)

In the third line, the integral over the reference space \( R \) of the total derivative in the second term is identical to zero, while the integral over \( r \) in the first term excludes the volume of the body, so that the boundary term remains, which is minus the force acting on the body. Therefore, the sum of the quasi-momentum of the fluid and the momentum of the body is conserved in an ideal fluid. That explains the surprising effect of acceleration of a shrinking moving body. Indeed, when the induced mass and the quasi-momentum of the fluid decrease then the body momentum must increase. Swimmers can get extra acceleration proportional to minus the time derivative of the added mass by a rapid decrease of the cross-section. Breaststroke kick-glide transition is better be fast. Octopus is particularly effective in rapid shape-change during acceleration.

This quasi-momentum (1.42) is defined for any flow. For a potential incompressible flow, one can obtain quasi-momentum simply integrating the potential over the body surface: \( K = -\int \rho \phi \, df \). Indeed, consider a very short and strong pulse of pressure needed to bring the body from rest into motion, formally \( p \propto b(t) \). During the pulse, the body doesn’t move, so its position and surface are well-defined. In the Bernoulli relation (1.23) one can then neglect the \( v^2 \) term:

\[ \frac{\partial \phi}{\partial t} = -\frac{\nu^2}{2} - \frac{p}{\rho} \approx -\frac{p}{\rho}. \] (1.45)

Integrating the relation \(-\rho \phi = \int p(t) \, dt\) over the body surface we get minus the change of the body momentum, i.e. the quasi-momentum of the fluid. For example, integrating \( \phi = -R^3 u \cos \theta / 2r^2 \) over the sphere we get

\[ K_x = -\int \rho \phi \cos \theta \, df = 2\pi \rho R^3 u \int_1^1 \cos^2 \theta \, d\cos \theta = 2\pi \rho R^3 u / 3, \]

as expected. The difference between momentum and quasi-momentum can be related to the momentum flux across the infinite surface due to pressure, which decreases as \( r^{-2} \) for a potential flow.

The quasi-momentum of the fluid is related to the body velocity via the induced mass, \( K_i = m_{ik} u_k \), so that one can use (1.42) to evaluate \( m_{ik} \). For this, one needs to solve the Lagrangian equation of motion \( \dot{r} = v(r, t) \); then one can show that the induced mass can be associated with the displacement of the fluid.
after the body pass. That would be wrong to think that in moving towards the right the body would displace fluid in front and leaves vacant space behind, so that there should be a net reflux towards the left in compensation. Let us show that net displacement is always in the direction of the motion. Consider an arbitrary potential flow shifting with a constant speed, \( \phi(x - ut, y, z) \), either due to moving body or propagating wave. To express the fluid displacement via the Lagrangian integral, we write

\[
v_x = \frac{\partial \phi}{\partial x} = -\frac{1}{u} \frac{\partial \phi}{\partial t} = \frac{1}{u} \left( |\nabla \phi|^2 - \frac{d\phi}{dt} \right)
\]

and substitute

\[
x(t) - x(0) = \int_0^t v_x(x(t'), y, z) \, dt' = \frac{1}{u} \int_0^t |\nabla \phi|^2 \, dt - \frac{1}{u} \phi|_0^t. \tag{1.46}
\]

We now choose \( t \) such that the last term is zero (period for the wave or infinity for passing body) and see that the displacement is always positive.

The body pushes the fluid in front and pulls the fluid behind, while the fluid on the sides moves opposite to the body. Indeed, the potential is \( \phi \propto -x/r^d \) (where \( d = 3 \) for a sphere, \( d = 2 \) for a cylinder), so that the horizontal component of the fluid velocity, \( v_x \propto d \cos^2 \theta - 1 \), changes sign when \( \cos^2 \theta = 1/d \). As a result, every particle makes a loop, as shown in Figure 1.9. Note the striking difference between the particle trajectories and instantaneous streamlines (see also Exercise 1.6). The permanently displaced mass enclosed between the broken lines (or swept out by any material surface spanning the fluid domain and lying across the direction of motion) is in fact the induced mass itself (Darwin 1953).
1.4 Viscosity

Let us summarize the previous section: neglecting tangential forces (i.e. internal friction) we were able to describe the inertial reaction of the fluid to the body acceleration (quantified by the induced mass). For a motion with a constant speed, we failed to find any force, including the force perpendicular to $u$ called lift. If that were true, flying would be impossible. Physical intuition suggests that the resistance force opposite to $u$ called drag must be given by the amount of momentum transferred to the fluid in front of the body per unit time (Newton 1687). Multiplying the momentum density $\rho u$ by the volume, which is the area $R^2$ times the velocity $u$ we obtain:

$$F = C R^2 \rho u^2,$$

where $C$ is some order-of-unity dimensionless constant (called the drag coefficient) depending on the body shape. This is the correct estimate for the resistance force in the limit of vanishing internal friction. To get a feeling for it, estimate that riding a bike with the speed $4 \text{ m/s} = 14.4 \text{ km/h}$ a body of cross-section $0.75 \text{ m}^2$ meets air drag approximately $12N$. A bagel without cream cheese has about $200 \text{kCal}$, which takes a ride of some $7 \text{ km}$ to burn.\textsuperscript{12}

Even though (1.47) does not contain viscosity, I don’t know any other way to show its validity but to introduce viscosity first and then consider the limit when it vanishes. That limit is quite non-trivial: even an arbitrary small friction makes an infinite region of the flow (called the wake) very much different from the potential flow described in the previous Section. Introducing viscosity and describing the wake will take this section and the next one.

1.4.1 Reversibility paradox

Let us consider the absence of resistance in a more general way. We have made five assumptions about the flow: that it is incompressible, irrotational, inviscid (ideal), infinite and steady. The last can always be approached with sufficient precision by waiting long enough (after the body has passed a distance equivalent to a few times its size is usually enough). An irrotational flow of an incompressible fluid is completely determined by the instantaneous body position and velocity. When the body moves with a constant velocity, the flow pattern moves along without changing its form; neither quasi-momentum nor kinetic energy of the fluid change so there are no work-doing forces acting between the fluid and the body. If the fluid is finite, that is has a surface, a finite drag arises due to surface waves. If the surface is far away from the body, that drag is negligible.
Could it be that an account of compressibility gives a finite drag for a steady flow, say, due to sound waves carrying energy away? This is not the case, as follows from the reversibility of the continuity and Euler equations: the reverse of the flow [defined as \( w(r, t) = -v(r, -t) \)] is also a solution with the velocity at infinity \( u \) instead of \(-u\) but with the same pressure and density fields. For the steady flow, defined by the boundary problem

\[
\text{div} \rho v = 0, \quad v_n = 0 \text{ (on the body surface)}, \quad v \to -u \text{ at infinity},
\]

\[
\frac{v^2}{2} + \int \frac{dp}{\rho(p)} = \text{const}.
\]

the reverse flow \( w(r) = -v(r) \) has the same pressure field, so it must give the same drag force on the body. Since the drag is supposed to change sign when the direction of motion is reversed, the drag is zero in an ideal irrotational flow. For the particular case of a body with a symmetry, reversibility gives d’Alembert’s paradox. For example, if there is a central symmetry, then the pressure on the symmetrical surface elements is the same and the resulting force is a pure couple.\(^{13}\)

Reversibility paradox teaches us that drag can only come from a force, which is odd with respect to velocity. That requires going beyond an ideal fluid where the pressure \( p \propto v^2 \) is even. Such drag force can come only from friction which owes its existence to molecular motion in the fluid. To find our way out of the paradoxes of ideal flows towards a real world requires considering internal friction, that is viscosity. Below we shall show that friction provides for drag and lift acting on a body moving through the fluid.

### 1.4.2 Viscous stress tensor

To describe normal and tangential forces we need now to specify both the orientation of the force component and the orientation of the surface on which it acts. It requires a tensor of the second rank. We define the stress tensor \( \sigma_{ij} \)
1.4 Viscosity

with the \( ij \) element equal to the \( i \) component of the force acting on a unit area perpendicular to the \( j \) direction. Without a flow, we have only pressure providing the diagonal components, which are normal stresses equal to each other by Pascal’s law. Nonuniform flow causes internal friction which changes the stress tensor: \( \sigma_{ik} = -p\delta_{ik} + \sigma'_{ik} \) (here the stress is applied to the fluid element under consideration so that the pressure is negative). This changes the momentum flux, \( \Pi_{ik} = p\delta_{ik} - \sigma'_i + \rho v_i v_k \), as well as the Euler equation: \( \partial \rho v_i / \partial t = -\partial \Pi_{ik} / \partial x_k \).

\[ \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \]

Figure 1.10 Diagonal and non-diagonal components of the stress tensor.

To avoid infinite rotational accelerations, the stress tensor must be symmetric: \( \sigma_{ij} = \sigma_{ji} \). Indeed, consider the moment of force (with respect to the axis at the upper right corner) acting on an infinitesimal element with the sizes \( \delta x, \delta y, \delta z \):

\[ (\sigma_{xz} - \sigma_{zx}) \delta x \delta y \delta z = \rho \delta x \delta y \delta z \left[ (\delta x)^2 + (\delta z)^2 \right] \frac{\partial \Omega}{\partial t}. \]

We see that to avoid \( \partial \Omega / \partial t \to \infty \) as \( (\delta x)^2 + (\delta z)^2 \to 0 \) we must assume that \( \sigma_{xz} = \sigma_{zx} \).
To connect the frictional stress tensor \( \sigma' \) and the velocity \( \mathbf{v}(\mathbf{r}) \), note that \( \sigma' = 0 \) for a uniform flow, so \( \sigma' \) must depend on the velocity spatial derivatives. This simple statement deserves reflection. Molecular motion is expected to provide the flux of any quantity (like heat, concentration of pollutant etc) determined by the gradient of the quantity. The stress tensor is a momentum flux, yet its frictional part due to molecular motion must be determined by the gradient of velocity rather than the gradient of momentum. Uniformly moving fluid with a non-uniform density does not experience internal friction. That is we assume the medium to be in thermal equilibrium so that there is no fluxes without a flow.

When the derivatives \( \partial v_i / \partial x_k \) are small compared with the velocity changes on a molecular level, one may assume that the stress tensor is linearly proportional to the tensor of velocity derivatives (Newton 1687). Fluids with this property are called Newtonian. Non-Newtonian fluids are those of elaborate molecular structure (e.g. with long molecular chains, like polymers), where the relation may be non-linear already for moderate strains, and rubber-like liquids, where the stress depends on the fluid’s history. For Newtonian fluids, to relate linearly two second-rank tensors, \( \sigma^i_j \) and \( \partial v_l / \partial x_l \), one generally needs a tensor of the fourth rank:

\[
\sigma^i_j = \eta \delta_{ik} \delta_{jl} \partial v_k / \partial x_l + \mu \delta_{ij} \partial v_l / \partial x_l.
\]  
(1.48)

Note that in a non-uniform flow, viscosity not only provides tangential components of the stress tensor but also breaks equality of the diagonal components.

Dimensionally \( \eta = \mu = g \text{ cm}^{-1} \text{ s}^{-1} \). To establish the sign of \( \eta \), consider a simple shear flow (shown in the figure) and recall that the stress is applied to the fluid. The stress component \( \sigma_{xz} = \eta \partial v_x / \partial z \) is the x component of the force by which an upper layer of the fluid acts on the lower layer so it must be positive, which requires \( \eta > 0 \).
1.4 Viscosity

\[ \sigma_{xz} = \eta \frac{dv}{dz} \]

1.4.3 Navier–Stokes equation

Now we substitute \( \sigma' \) into the Euler equation

\[ \rho \left( \frac{\partial v_i}{\partial t} + v_k \frac{\partial v_i}{\partial x_k} \right) = -\frac{\partial}{\partial x_k} \left[ p \delta_{ik} - \eta \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right) - \mu \delta_{ik} \frac{\partial v_l}{\partial x_l} \right] \]. \ (1.49)

The viscosity is determined by the thermodynamic state of the system, that is by \( p, \rho \). When \( p, \rho \) depend on coordinates, so must \( \eta(p, \rho) \) and \( \mu(p, \rho) \). However, we consistently assume that the variations of \( p, \rho \) are small and put \( \eta, \mu \) constant. In this way we get the famous Navier–Stokes equation\(^{15}\):

\[ \rho \frac{dv}{dt} = -\nabla p + \eta \Delta v + (\eta + \mu) \nabla \text{div} v. \] \ (1.50)

Apart from the case of rarefied gases we cannot derive this equation consistently from kinetics. This means only that we generally cannot quantitatively relate \( \eta \) and \( \mu \) to the properties of the material, the form of the equation is beyond doubt. One can estimate the viscosity of a gas, saying that the flux of molecules with thermal velocity \( v_T \) through the plane (perpendicular to the velocity gradient) is \( n v_T \) and that the molecules come from a layer comparable to the mean free path \( l \) and have velocity difference \( l \Delta u \), which causes momentum flux \( mn v_T l = \rho v_T l \). Apparently, this estimate makes sense only when the velocity changes on the scale far exceeding the mean free path.\(^{16}\) The mean free path can be expressed as \( l = 1/\sigma \), where \( \sigma \) is the scattering cross-section. The viscosity of gases, \( \eta \approx m v_T / \sigma \), is then independent of density and pressure at a fixed temperature. The thermal velocity grows with the temperature so that viscosity increases with temperature for gases at constant density. The scattering cross-section is determined by the strength of interaction between molecules: the stronger the interaction, the larger is \( \sigma \), the shorter is \( l \) and the smaller is the viscosity of the gas. Strongly interacting quark-gluon plasma (in colliders and right after the big bang) is an almost ideal fluid. We also define kinematic viscosity \( \nu = \eta / \rho \), which is estimated as \( \nu \approx v_T l \).
The estimate $\nu \simeq v_T l$ does not make much sense for a liquid, where a shear stress is provided by intermolecular forces rather than by molecules coming from place to place. Stronger interaction then means larger viscosity for liquids. As temperature increases, molecules move faster and interact weaker, decreasing the stress, so that $\nu$ generally decreases with temperature for liquids (as any cook will readily confirm). Cooling liquid down to the freezing point, one increases viscosity to infinity.

Comparing most common liquid and gas, one finds that at room temperature air has $\nu = 0.15 \text{ cm}^2 \text{s}^{-1}$ and is kinematically 15 times more viscous than water, which has $\nu = 0.01 \text{ cm}^2 \text{s}^{-1}$. That means that if one creates a localized vortex, it diffuses its vorticity 15 times faster in the air than in the water (see also Problems 1.9 and 1.15).

The Navier–Stokes equation has higher-order spatial derivatives (second-order) than the Euler equation so that we need more boundary conditions. Since we accounted (in the first non-vanishing approximation) for the forces between fluid layers, we also have to account for the forces of molecular attraction between a viscous fluid and a solid body surface. Such a force makes the layer of adjacent fluid to stick to the surface: $v = 0$ on the surface (not just $v_n = 0$ as for the Euler equation). When liquid is in contact with a vacuum or rarefied gas, the boundary cannot support the viscous stress and the boundary condition is no-stress: $\partial v_l / \partial x_n = 0$. The solutions of the Euler equation generally satisfy neither no-slip nor no-stress boundary condition. This means that even a very small viscosity must play a role near a surface.

Viscosity adds an extra term to the momentum flux, but (1.49) and (1.50) still have the form of a continuity equation that conserves total momentum. However, viscous friction between fluid layers necessarily leads to some energy dissipation. Consider, for instance, a viscous incompressible fluid with $\text{div } v = 0$ and calculate the time derivative of the energy at a point:

$$\frac{\rho}{2} \frac{\partial v^2}{\partial t} = -\rho v \cdot (v \nabla) v - v \cdot \nabla p + v_i \frac{\partial \sigma'_i k}{\partial x_k}$$

$$= -\text{div} \left[ \rho v \left( \frac{v^2}{2} + \frac{p}{\rho} \right) - (v \cdot \sigma') \right] - \sigma'_{ik} \frac{\partial v_i}{\partial x_k}. \quad (1.51)$$

The presence of viscosity results in the momentum flux $\sigma'$, which is accompanied by the energy transfer, $v \cdot \sigma'$, and the energy dissipation described by the last term. Because of this last term, this equation does not have the form of a continuity equation and the total energy integral is not conserved. Indeed, after
1.4 Viscosity

The integration over the whole volume,

\[
\frac{dE}{dt} = -\int \sigma'_{ik} \frac{\partial v_i}{\partial x_k} dV = -\eta \int \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 dV
\]

\[= -\eta \int \omega^2 dV < 0 . \tag{1.52} \]

The last equality here follows from \( \omega^2 = (\epsilon_{ijk} \partial_j v_k)^2 = (\partial_j v_k)^2 - \partial_k (\partial_j v_j v_k), \) which is true by virtue of \( \epsilon_{ijk} \epsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \) and \( \partial_i v_i = 0. \) Since \( \eta > 0 \) then viscosity indeed dissipates energy. We see that the dissipation is related to vorticity in an incompressible flow, since the Laplacian of the potential velocity is zero.

The Navier–Stokes equation is a non-linear partial differential equation of the second order. Not many steady solutions are known. It is particularly easy to find solutions for the geometry where \( (v \cdot \nabla) v = 0 \) and the equation is effectively linear. In particular, symmetry may prescribe that the velocity does not change along itself. One example is the flow along an inclined plane as a model for a river.

![River flow diagram](image)

Everything depends only on \( z. \) The stationary Navier–Stokes equation takes the form

\[-\nabla p + \eta \Delta v + \rho g = 0\]

with \( z \) and \( x \) projections, respectively,

\[
\frac{dp}{dz} + \rho g \cos \alpha = 0 , \quad \frac{d^2 v}{dz^2} + \rho g \sin \alpha = 0 . \tag{1.53} \]

The boundary condition on the bottom is \( v(0) = 0. \) On the surface, the boundary condition is that the stress should be normal and balance the pressure: \( \sigma_{zx}(h) = \eta dv(h)/dz = 0 \) and \( \sigma_{zz}(h) = -p(h) = -p_0. \) The solution is simple:

\[p(z) = p_0 + \rho g (h - z) \cos \alpha , \quad v(z) = \frac{\rho g \sin \alpha}{2\eta} z(2h - z) . \tag{1.54} \]
Let us see how it corresponds to reality. Take water with the kinematic viscosity \( \nu = \eta/\rho = 10^{-2} \text{cm}^2 \text{s}^{-1} \). For a rain puddle with thickness \( h = 1 \text{ mm} \) on a slope \( \alpha \sim 10^{-2} \) we get a reasonable estimate \( v \sim 5 \text{ cm s}^{-1} \). For slow plain rivers (like the Nile or the Volga) with \( h \simeq 10 \text{ m} \) and \( \alpha \simeq 0.3 \text{ km}/3000 \text{ km} \simeq 10^{-4} \) one gets \( v(h) \simeq 100 \text{ km s}^{-1} \) which is evidently impossible (the resolution of this dramatic discrepancy is that real rivers are turbulent, as discussed in Section ??). What distinguishes puddle and river, why are they not similar? To answer this question, we need to characterize flows by a dimensionless parameter.

### 1.4.4 Law of similarity

One can obtain some important conclusions about flows from a dimensional analysis. Consider a steady incompressible flow past a body described by the equation

\[
(\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla(p/\rho) + \nu \nabla^2 \mathbf{v}
\]

and by the boundary conditions \( \mathbf{v}(\infty) = \mathbf{u} \) and \( \mathbf{v} = 0 \) on the surface of a body of size \( L \). For a given body shape, both \( \mathbf{v} \) and \( p/\rho \) are functions of coordinates \( \mathbf{r} \) and three variables, \( \mathbf{u}, \nu, L \). Out of the latter, one can form only one dimensionless quantity, called the Reynolds number

\[
Re = \frac{uL}{\nu}.
\] (1.55)

This is the most important parameter in this book, since it determines the ratio of the non-linear (inertial) term \( (\mathbf{v} \cdot \nabla)\mathbf{v} \) to the viscous friction term \( \nu \nabla^2 \mathbf{v} \). Indeed, in the dimensionless variables \( \mathbf{v}/u, \mathbf{r}/L \) the incompressible Navier-Stokes equation takes the form

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \frac{p}{\rho} + \frac{1}{Re} \nabla^2 \mathbf{v}.
\] (1.56)

Since the kinematic viscosity is the thermal velocity times the mean free path, the Reynolds number is

\[
Re = \frac{uL}{\nu_T l}.
\]

We see that within the hydrodynamic limit \( (L \gg l) \), \( Re \) can be both large and small depending on the ratio \( u/\nu_T \simeq u/c \).

Dimensionless velocity must be a function of dimensionless variables: \( \mathbf{v} = u \Gamma(\mathbf{r}/L, Re) \) it is a unit-free relation. Flows that correspond to the same \( Re \) can be obtained from one another simply by changing the units of \( v \) and \( r \); such flows
1.5 Viscosity

are called similar (Reynolds 1883). In the same way, \( p/\rho = u^2\psi(r/L, Re) \). For a quantity independent of coordinates, only some function of \( Re \) is unknown – the drag or lift force, for instance, must be \( F = \rho u^2 L^2 f(Re) \). This law of similarity is exploited in modelling: to measure, say, the drag on a ship that one is designing, one can build a smaller model yet pull it faster through the fluid (or use a less viscous fluid).

The Reynolds number, as a ratio of inertia to friction, makes sense for all types of flow as long as \( u \) is some characteristic velocity and \( L \) is a scale of the velocity change. For the inclined plane flow (1.54), the non-linear term (and the Reynolds number) is zero since \( v \perp \nabla v \). How much does one need to perturb this alignment to make \( Re \simeq 1 \)? Such perturbations always exist in reality where the bottom is never perfectly flat. One may think that bottom imperfections are more important for a shallow paddle than for a deep river. It is the other way around. Denoting \( \pi/2 - \beta \) the angle between \( v \) and \( \nabla v \) we get

\[
Re(\beta) = v(h)h\beta/v \simeq \alpha \beta h^3/v^2. 
\]

For a puddle, \( Re(\beta) \simeq 50\beta \) while for a river \( Re(\beta) \simeq 10^{12}\beta \). It is then clear that the (so-called laminar) solution (1.54) may make sense for a puddle, but for a river it must be distorted by even tiny bottom imperfections, see Figure 1.11).

Gravity brings another dimensionless parameter, the Froude number \( Fr = u^2/Lg \); the flows are similar for the same \( Re \) and \( Fr \). Such parameters (changes of which bring qualitative changes in the regime even for fixed geometry and boundary conditions) are called control parameters.\(^{18}\)

The law of similarity is a particular case of the so-called \( \pi \)-theorem: Assume that among all \( m \) variables \( \{b_1, \ldots, b_m\} \) we have only \( k \leq m \) dimensionally independent quantities – this means that the dimensionalities \( b_{k+1}, \ldots, b_m \) could be expressed via \( b_1, \ldots, b_k \) like \( b_{k+i} = \prod_{l=1}^{i} b_l^{\beta_l} \). Then all dimensionless quantities can be expressed in terms of \( m-k \) dimensionless variables \( \pi_1 = b_{k+1}/\prod_{l=1}^{i} b_l^{\beta_l}, \ldots, \pi_{m-k} = b_m/\prod_{l=1}^{i} b_l^{\beta_l} \). For example, the three above quantities \( u, v, L \) have two independent dimensionalities, \( cm \) and \( sec \), which allows one to introduce the single dimensionless parameter, the Reynolds number.

\[ \text{Figure 1.11 The non-flat bottom of the river bed makes the velocity of the river change along itself, which leads to a non-zero inertial term (v \cdot \nabla v) in the Navier–Stokes equation.} \]
1.5 Stokes flow and the wake

We now return to the flow past a body armed with the knowledge of internal friction. Unfortunately, the Navier–Stokes equation is a non-linear partial differential equation which we cannot solve in a closed analytical form even for a flow around a sphere. We shall therefore proceed in the way that physicists often do: solve a limiting case of very small Reynolds numbers and then use this solution to understand higher-Re flow. Remember that in Section 1.3 we failed spectacularly to describe high-Re flow as an ideal fluid. This time we shall realize, with the help of qualitative arguments and experimental data, that when viscosity becomes very small its effect stays finite. On the way we shall learn new notions of a boundary layer and a separation phenomenon. Our reward will be the resolution of paradoxes and the formulae for drag and lift.

1.5.1 Slow motion

Consider such a slow motion of a body through the fluid that the Reynolds number, \( Re = uR/\nu \), is small. This means that we can neglect inertia. Indeed, if we stop pushing the body, friction stops it after a time of order \( R^2/\nu \), so that inertia moves it by the distance of order \( uR^2/\nu = R \cdot Re \), which is much less than the body size \( R \). Formally, neglecting inertia means omitting the non-linear term \( (\mathbf{v} \cdot \nabla)\mathbf{v} \) in the Navier–Stokes equation. This makes our problem linear so that the fluid velocity is proportional to the body velocity: \( \mathbf{v} \propto u \). The viscous stress (1.48) and the pressure are also linear in \( u \) and so must be the drag force:

\[
F = \int \sigma \, d\mathbf{f} \simeq \int \mathbf{f} \eta u/R \simeq 4\pi R^2 \eta u/R = 4\pi \eta u R.
\]

This crude estimate coincides with the true answer given later by (1.60) up to the dimensionless factor \( 3/2 \). Linear proportionality between the force and the velocity makes the low-Reynolds flow an Aristotelean world.

Now, if you wish to know what force would move a body with \( Re \simeq 1 \) (or \( 1/6\pi \) for a sphere), you find amazingly that such a force, \( F \sim \eta^2/\rho \), does not depend on the body size (that is, it is the same for a bacterium and a ship). For water, \( \eta^2/\rho \simeq 10^{-4} \text{dyn} = 10^{-9} \text{N} \).

If we also assume than no fast-changing forces is applied to the fluid then the whole inertia term, \( \rho d\mathbf{v}/dt = \rho \partial \mathbf{v}/\partial t + (\mathbf{v} \nabla)\mathbf{v} \), can be neglected:

\[
\partial \mathbf{v}/\partial t \simeq (\mathbf{v} \nabla)\mathbf{v} \simeq u^2/L \ll v \Delta v \simeq vu/L^2.
\]
In this case, the Navier–Stokes equation (1.50) turns into the Stokes equation:

\[ \eta \Delta \mathbf{v} = \nabla p. \]  

(1.57)

One can also say that this equation describes the flow of a massless fluid, sometimes called creeping flow. In particular, motion on microscopic and nanoscopic scales in fluids usually corresponds to very low Reynolds numbers and is described by the Stokes equation. Swimming at low \( Re \) is very different from pushing water backwards as we do at finite \( Re \). One defines swimming as changing shape in a periodic way to move. First, there is no inertia at low \( Re \) so that momentum diffuses instantly through the fluid. Therefore, it does not matter how quickly or slowly we change the shape. What matters is the shape changes itself, i.e. low-\( Re \) swimming is purely geometrical. Second, linearity means that simply retracing the changes back (by inverting the forces, i.e. the pressure gradients) we just retrace the motion. One thus needs to change a shape periodically but in a time-irreversible way, that is to have a cycle in a configuration space. Micro-organisms do that by sending progressive waves along their surfaces. Every point of a surface may move time-reversibly (even in straight lines); the time direction is encoded in the phase shift between different points. For example, a spermatozoid swims by sending helical waves down its tail.\(^{19}\) See Exercise 1.10 for another example.

Probably, the simplest way to find solutions of the Stokes equation (1.57) is to reduce it again to the Laplace equation. Since creeping flows practically always can be treated as incompressible, we can use \( \text{div} \mathbf{v} = 0 \) to apply the operator \( \text{div} \) to (1.57) and obtain \( \Delta p = 0 \). Let us solve this equation for the flow around a sphere when the only vector in the problem is the velocity \( \mathbf{u} \). The respective solution of the Laplace equation for a scalar (this time it is pressure rather than potential) is again a dipole:

\[ p - p_0 = \frac{c(u \cdot n)}{r^2}. \]

Indeed, positive and negative pressure variations must have the same magnitude on respective points of the sphere. We now differentiate it and substitute into (1.57). We can obtain the partial solution of the resulting equation with the radial velocity perturbation decaying as an inverse distance: \( \mathbf{v} - \mathbf{u} = c \mathbf{n}(\mathbf{u} \cdot \mathbf{n})/2\eta r \). To make it incompressible we need to add a solution of the homogeneous equation: \( \mathbf{v} - \mathbf{u} = c \mathbf{u} + \mathbf{n}(\mathbf{u} \cdot \mathbf{n})/2\eta r \). This solution, however, cannot satisfy the boundary condition on the sphere surface. For that we need to add yet another solution of
the Laplace equation that is the potential part:

$$
\mathbf{v} = \mathbf{u} + c \frac{\mathbf{u} + \mathbf{n}(\mathbf{u} \cdot \mathbf{n})}{2\eta r} + b \frac{3\mathbf{n}(\mathbf{u} \cdot \mathbf{n}) - \mathbf{u}}{r^3}. \tag{1.58}
$$

The boundary condition \( \mathbf{v}(R) = 0 \) gives \( \mathbf{u} \) component \( 1 + c/2\eta R - b/R^3 = 0 \) and \( \mathbf{n} \) component \( c/2\eta R + 3b/R^3 = 0 \) so that \( c = -6\eta R/4 \) and \( b = R^3/4 \). In spherical components

\[ 
\begin{align*}
\nu_r &= u \cos \theta \left( 1 - \frac{3R}{2r} + \frac{R^3}{2r^3} \right), \\
\nu_\theta &= -u \sin \theta \left( 1 - \frac{3R}{4r} - \frac{R^3}{4r^3} \right). \tag{1.59}
\end{align*}
\]

Sanity check confirms that \( c < 0 \), that is the pressure is larger upstream so that fluid flows down the pressure gradient. The vorticity too satisfies the Laplace equation and is a dipole field:

$$
\Delta \text{curl} \mathbf{v} = \Delta \omega = 0 \Rightarrow \omega = c' \frac{\mathbf{u} \times \mathbf{n}}{r^2}
$$

with \( c' = -3R/2 \) from \( \nabla p = \eta \Delta \mathbf{v} = -\eta \text{curl} \omega \).

**Stokes' formula** for the drag. The force acting on a unit surface is the momentum flux through it. On a solid surface \( \mathbf{v} = 0 \) and \( F_i = -\sigma_{ik}n_k = pn_i - \sigma'_{ik}n_k \). In our case, the only non-zero component is along \( \mathbf{u} \):

$$
F_x = \int (-p \cos \theta + \sigma'_{rr} \cos \theta - \sigma'_{r\theta} \sin \theta) \, df = (3\eta u/2R) \int df = 6\pi R\eta u. \tag{1.60}
$$
Here, we substituted $\sigma''_{rr} = 2\eta \partial v_r / \partial r = 0$ at $r = R$ and

$$p(R) = \frac{-3\eta u}{2R} \cos \theta,$$

$$\sigma''_{r\theta}(R) = \eta \left( \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) = \frac{-3\eta u}{2R} \sin \theta.$$

The viscous force is tangential while the pressure force is normal to the surface. The vertical components of the forces cancel each other at every point — the sphere pushes fluid strictly forward and the force is purely horizontal. The viscous and pressure contributions sum into the horizontal force $3\eta u/2R$, which is independent of $\theta$, i.e. the same for all points on the sphere. The force is determined by the dynamic viscosity and independent of the fluid density. The density will enter via the dimensionless Reynolds number $Re = \rho u R / \eta$ when we account for fluid inertia. The formula (1.60) is called Stokes’ law; it works well until $Re \simeq 0.5$.

We can now generalize the equation of motion for particles in a flow, (1.37), by including viscous force. Considering for simplicity a heavy spherical particle with the radius $a$ and neglecting fluid inertia, we relate the particle velocity $u$ to the fluid velocity $v$:

$$\frac{du}{dt} = v - u \tau, \quad \tau = \frac{2\rho_0 a^2}{9\rho v}.$$

(1.61)

Formal solution of this equation,

$$u(t) = \int_{-\infty}^{t} e^{(t-\tau')/\tau} v(t') \, dt',$$

(1.62)

is expressed via the fluid velocity in the reference frame co-moving with the particle. The solution shows that the particle motion is retarded by the response time $\tau$ which is called Stokes time. For millimeter-sized rain droplets $\tau \simeq 10$ sec, while for micron-size cloud droplets it is million times less.

The solution of the boundary value problem for the Laplace equation is unique. So if we require the pressure gradient to go to zero at infinity, then a source of any form having constant pressure on the surface produces constant pressure in the whole space. Similarly, since the vorticity must tend to zero at infinity, it can be nonzero in space only if it is nonzero on the source surface. In particular, a point source produces a radial flow having a constant pressure and zero vorticity (see Exercise 1.16). By superposition this is also true for an arbitrary combination of point sources and sinks since creeping flows are linear.
In two dimensions, pressure and vorticity can be treated as, respectively, real and imaginary part of an analytic function, which brings power of complex analysis to the problem. In particular, zero pressure gradient means zero vorticity and vice versa.

1.5.2 The boundary layer and the separation phenomenon

Another thing named after Stokes is the paradox - no finite solution of (1.57) exists for flow past a body in two dimensions. Indeed, the dipole fields in two dimensions must decay as $1/r$. When pressure and vorticity decay as $1/r$, the velocity logarithmically grows with $r$ (rather than saturates as in 3d). Stokes himself believed that moving a long cylinder through a very viscous fluid one continually 'increases the quantity of fluid which it carries with it'. Rayleigh later pointed out that for the Stokes flow the viscous term decays with the distance faster than the inertial term, so that the latter must be taken into account sufficiently far from the body. Indeed, the assumption of small Reynolds number requires in any dimensionality

$$v \nabla v \simeq u^2 R/r^2 \ll v \Delta v \simeq v u R^2/r^3,$$  \hspace{1cm} (1.63)

so that the Stokes equation is valid for $r \ll v/u$. One can call $v/u$ the width of the viscous boundary layer. The existence of the boundary layer resolves Stokes paradox: inertia must stop the growth of velocity perturbation at $r \simeq v/u$. If we put a cylinder in a uniform flow having velocity $u$ at infinity, then $v(r = v/u) \simeq u$, and the steady solution at $R \leq r < v/u$ is as follows:

$$v(r) \simeq u \log(r/R) \log(v/u R).$$  \hspace{1cm} (1.64)

That helps to understand another 2d peculiarity: the drag on a cylinder is a force per unit length $f$, for which the only dimensionally possible combination linear in viscosity is $\rho u v$, absurdly independent of the size $R$. We expect the drag to increase with $R$ (and turn to zero as $R \to 0$), so that the dimensionless drag $f/\rho u v$ must go to zero when $Re = u R/v \to 0$. Indeed, the friction force is determined by the fluid velocity gradient on the cylinder, which according to (1.64) depends logarithmically on the size: $f \simeq \rho u v / \log Re^{-1}$, that is the drag is logarithmically suppressed at small $Re$. Nonlinear dependence of the force on the viscosity in two dimensions shows that no matter how small is $Re$, one cannot neglect inertia and consider fluid massless as long as any fluid element is infinite in the third direction.
1.5 Stokes flow and the wake

In both two and three dimensions the Stokes flows are realized inside the boundary layer under the assumption that the size of the body is much less than the width of the layer. So what is the flow outside the viscous boundary layer, that is for \( r > v/\nu \)? Is it potential? The answer is ‘yes’ only for very small \( Re \). For finite \( Re \), there is an infinite region (called the wake) behind the body where it is impossible to neglect viscosity whatever the distance from the body. This is because viscosity produces vorticity in the boundary layer:

\[
\omega_z = \frac{dv_x}{dz}
\]

At small \( Re \), the process that dominates the flow is vorticity diffusion away from the body caused by friction. The Stokes approximation, \( \omega \propto u \times n/r^2 \), corresponds to symmetrical diffusion of vorticity in all directions. In particular, the flow has a left–right (fore-and-aft) symmetry. For finite \( Re \), vorticity production by friction is accompanied by inertial vorticity advection; it is then intuitively clear that the flow upstream and downstream from the body must be different since the body leaves vorticity behind it. Indeed, when the boundary layer is comparable to the body size or less, the fluid particles enter the layer with zero vorticity but leave it with a non-zero vorticity, which is then carried further since inertia dominates friction outside. Therefore, there should exist some downstream region reached by fluid particles which move along streamlines passing through the boundary layer. The flow in this region (wake) is essentially rotational. On the other hand, streamlines that do not pass through the boundary layer correspond to almost potential motion.

Let us describe qualitatively how the wake arises. The phenomenon called separation is responsible for wake creation (Prandtl 1905). Consider, for instance, the flow around a cylinder, shown in Figure 1.12. The ideal fluid flow is symmetrical with respect to the plane AB. The point D is a stagnation point. On the upstream half DA, the fluid particles accelerate and the pressure

![Figure 1.12 Symmetric streamlines for an ideal flow (left) and appearance of separation and a recirculating vortex in a viscous fluid (right).](image-url)
Basic notions and steady flows

decreases according to the Bernoulli theorem. On the downstream part AC, the reverse happens, that is every particle moves against the pressure gradient. A small viscosity changes pressure only slightly across the boundary layer. Indeed, if the viscosity is small, the boundary layer is thin and can be considered locally flat. Denote $u$ the velocity right outside the boundary layer. In the boundary layer, at $z < u/v$, no-slip condition prescribes $v_z \simeq u^2 z/v$ and $\partial v_z/\partial x \simeq u^2 z/vR \simeq \partial v_z/\partial z$. The normal velocity is then $v_z \simeq u^2 z/vR$, which gives the pressure gradient, $\partial p/\partial z = -\rho (v \nabla) v - \eta \Delta v_z \simeq \rho u^2 / R$, so that the pressure change across the layer is $\rho u^2 / Re$ that is small when $Re$ is large. In other words, the pressure inside the boundary layer is almost equal to that in the main stream, which is the pressure of the ideal fluid flow. But the velocities of the fluid particles that reach the points A and B are lower in a viscous fluid than in an ideal fluid because of viscous friction in the boundary layer. Then those particles have insufficient energy to overcome the pressure gradient downstream. The particle motion in the boundary layer is stopped by the pressure gradient before the point C is reached. The pressure gradient then becomes the force that accelerates the particles from the point C upwards, producing separation and a recirculating vortex. A similar mechanism is responsible for recirculating eddies in the corners shown at the end of Section 1.2.4.

Reversing the flow pattern of separation one obtains attachment: jets tend to attach to walls and merge with each other. Consider first a jet in an infinite fluid and denote the velocity along the jet $u$. The momentum flux through any section is the same: $\int u^2 \, df = \text{const}$. On the other hand, the energy flux, $\int u^3 \, df$, decreases along the jet owing to viscous friction. This means that the mass flux of the fluid, $\int u \, df$, must grow - a phenomenon known as entrainment. When the jet has a wall (or another jet) on one side, it draws into itself less fluid from this side and so inclines towards the wall until it is attached, as shown in the figure. The jet then can stay attached and follow the surface even when it is convex.
In particular, jet merging explains a cumulative effect of armour-piercing shells, which contain a conical void covered by a metal and surrounded by explosives. Explosion turns the metal into a fluid. Moreover, the pressure is so high that the tangential stresses can be neglected and the fluid flow is ideal, see Exercise 1.18, which moves towards the axis where it creates a cumulative jet with a high momentum density (Lavrent’ev 1947, Taylor 1948), see Figure 1.13 and Exercise 1.15. Similarly, if one creates a void in a liquid with, say, a raindrop or other falling object then the vertical momentum of the liquid that rushes to fill the void creates a jet, shown in Figure 1.14.

1.5.3 Flow transformations

Let us now use the case of the flow past a cylinder to describe briefly how the flow pattern changes as the Reynolds number goes from small to large. The flow is most symmetric for $Re \ll 1$ when it is steady and has an exact up–down symmetry and approximate (order $Re$) left–right symmetry. Separation of the boundary layer and the occurrence of eddies is a change of the flow topology; it occurs around $Re \approx 5$. The first loss of exact symmetries happens around $Re \approx 40$ when the flow becomes periodic in time. This happens because the recirculating eddies don’t have enough time to spread; they are being detached from the body and carried away by the flow as new eddies are generated. Periodic flow with shedding eddies has up–down and continuous time shift symmetries broken and replaced by a combined symmetry of up–down reflection and time shift for half a period. The shedding of eddies explains many surprising symmetry-breaking phenomena, like, for instance, an air bubble rising through water (or champagne) in a zigzag or spiral rather than a straight path. It also has a strong effect on swimming and flying at moderate $Re$; many fish, birds and hovering
Figure 1.13 Scheme of the flow of a cumulative jet in the reference frame moving with the cone.

Figure 1.14 Jet shooting out after the droplet fall. Upper image – beginning of the jet formation; lower image – jet formed. Photograph copyright: Sdtr, Rmarmion, www.dreamstime.com.
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Figure 1.15 Kármán vortex street behind a cylinder at $Re = 105$. Photograph by Sadatoshi Taneda, reproduced from *J. Phys. Soc. Japan*, 20, 1714 (1965).

Insects are able to exploit the pressure variation resulted from vortex formation and detachment. For flow past a body, vortex shedding results in a double train of vortices, called the Kármán vortex street\(^{24}\) behind the body, as shown in Figure 1.15. Kármán vortex street is responsible for many acoustic phenomena like the roar of propeller, sound caused by a wind rushing past a tree or the swish of a whip.

As the Reynolds number increases further, the vortices become unstable and produce an irregular turbulent motion downstream, as seen in Figure 1.16.\(^{25}\) That turbulence is three-dimensional, i.e. the translational invariance along the cylinder is broken as well. The higher $Re$, the closer to the body turbulence starts. At $Re \gtrsim 10^5$, the turbulence reaches the body, making the rear part of the boundary layer turbulent and bringing the so-called drag crisis (discovered by Eiffel in 1912 and explained by Prandtl in 1914): Since a turbulent boundary layer entrains more fluid from outside, has thus more momentum and separates later downstream than a laminar one, the wake area gets smaller and the drag is reduced. One can check that for $Re < 10^5$ a stick encounters more drag when moving through a still fluid than when kept still in a moving fluid (in the latter case the flow is usually turbulent before the stick so that the boundary layer is turbulent as well). Generations of scientists, starting from Leonardo Da Vinci, believed that the drag must be the same (despite experience telling otherwise) because of Galilean invariance, which, of course, is applicable only to an infinite uniform flow, not to real streams.

1.5.4 Drag and lift with a wake

We can now describe the way Nature resolves reversibility and d’Alembert’s paradoxes. As in Section 1.3, we consider the steady flow far from the body
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Figure 1.16 Flow past a cylinder at \( Re = 10^4 \). Photograph by Thomas Corke and Hassan Najib, reproduced from [27].

and relate it to the force acting on the body. The new experimental wisdom we now have is the existence of the wake (Figure 1.17). The flow is irrotational outside the boundary layer and the wake. First, we consider a laminar wake, i.e. assume \( v \ll u \) and \( \partial v/\partial t = 0 \); we shall show that the wake is always laminar far enough from the body. For a steady flow, it is convenient to relate the force to the momentum flux through a closed surface. For a dipole potential flow \( v \propto r^{-3} \) from Section 1.3, that flux was zero for a distant surface. Now the wake gives a finite contribution. The total momentum flux transported by the fluid through any closed surface is equal to the rate of momentum change, which is equal to the force acting on the body:

\[
F_i = \oint \sum_{ik} \Pi_{ik} \, df_k = \oint (p_0 + p') \delta_{ik} + \rho (u_i + v_i) (u_k + v_k) \, df_k (1.65)
\]

In the last line, the first integral vanishes because the surface is closed and the second one because of mass conservation: \( \rho \oint v_k \, df_k = 0 \). Far from the body
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$v \ll u$ and we neglect terms quadratic in $v$:

$$F_i \approx \left( \int \int_{X_0} - \int \int_X \right) (p' \delta_{ix} + \rho u v_i) dydz. \tag{1.66}$$

**Drag with a wake.** Consider the $x$ component of the force (1.66):

$$F_x = \left( \int \int_{X_0} - \int \int_X \right) (p' + \rho u v_x) dydz.$$

Outside the wake we have potential flow where the Bernoulli relation, $p + \rho |u + v|^2/2 = p_0 + \rho u^2/2$, gives $p' \approx -\rho u v_x$ so that the integral outside the wake vanishes. Inside the wake, the pressure is about the same (since it does not change across the almost straight streamlines, as we argued in Section 1.5.2) but the velocity perturbation $v_x$ is shown below to be much larger than outside, so that

$$F_x = -\rho u \int \int_{\text{wake}} v_x dydz. \tag{1.67}$$

Force is positive (directed to the right) since $v_x$ is negative. The integral in (1.67) is equal to the deficit of fluid flux $Q$ through the wake area (i.e. the difference between the flux with and without the body). That deficit is $x$-independent, which has dramatic consequences for the potential flow outside the wake because it has to compensate for the deficit. That means that the integral $\int v df$ outside the wake is also $r$-independent which requires $v \propto r^{-2}$. This corresponds to the potential flow with the source equal to the flow deficit: $\phi = Q/r$. It is analogous to a charge field in electrostatics. We had thrown away this source flow in Section 1.3 but now we see that it exceeds the dipole flow $\phi = A \cdot \nabla (1/r)$ (which we had without the wake) and dominates sufficiently far from the body.

The wake breaks the fore-and-aft symmetry and thus resolves the paradoxes, providing for a non-zero drag in the limit of vanishing viscosity. Now that we learnt that drag is related to the vorticity production by the body, we can appreciate how slender fish swim by passing a wave along its body. As explained in Section 1.2.4, small oscillations produce little vorticity, so the fish keeps the wave amplitude small over most of the body, increasing it towards the tail. It is important that the wake has an infinite length under stationary conditions, otherwise the body and the finite wake could be treated as a single entity and we are back to paradoxes. The behaviour of the drag coefficient $C(Re) = F/\rho u^2 R^2$ is shown in Figure 1.18. Notice the drag crisis, which gives the lowest $C$. To
understand why $C \rightarrow \text{const.}$ as $Re \rightarrow \infty$ and prove (1.47), one needs to go a long way, developing the theory of turbulence briefly described in the next chapter.

The lift is the force component of (1.66) perpendicular to $u$:

$$F_y = \rho u \left( \int_{X_0}^{\infty} - \int_{\infty}^{X_0} \right) v_y \, dy \, dz. \quad (1.68)$$

It is also determined by the wake – without the wake the flow is potential with $v_y = \partial \phi / \partial y$ and $v_z = \partial \phi / \partial z$ so that $\int v_y \, dy \, dz = \int v_z \, dy \, dz = 0$ since the potential is zero at infinity. We have seen in (1.34) that purely potential flow produces no lift. Without friction-caused separation, birds and planes would not be able to fly. For a wing, which is long in the $z$-direction, the lift force per unit length can be related to the velocity circulation around the wing. Indeed, adding and subtracting (vanishing) integrals of $v_x$ over two $y = \pm \text{const.}$ lines we turn (1.68) into

$$F_y = \rho u \oint \mathbf{v} \cdot d\mathbf{l}. \quad (1.69)$$

Circulation over the contour is equal to the vorticity flux through the contour, which is again due to the wake.

To fly effectively, wings must minimize the drag proportional to the wake area, keeping the lift proportional to the circulation. This is achieved by making wings slender with the thickness $h$ much less than the width $l$ and by making the leading edge smooth and the trailing edge sharp. One can often hear a simple explanation of the lift of the wing as being the result of $v_2 > v_1 \Rightarrow P_2 < P_1$. This is basically true and does not contradict the above argument.

![Figure 1.18 Sketch of the drag dependence on the Reynolds number.](image)
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The point is that the circulation over the closed contour ACDB is non-zero: \( v_2 l_2 > v_1 l_1 \). Indeed, the Bernoulli relation, \( P_2 - P_1 = \rho (v_1^2 - v_2^2)/2 = \rho (v_1 - v_2)(v_1 + v_2)/2 \approx \rho (v_1 - v_2)u \), gives an alternative derivation of the lift via circulation: \( \int (P_2 - P_1)dx \approx \rho u \int (v_1 - v_2)dx \). It would be wrong, however, to argue that \( v_2 > v_1 \) because \( l_2 > l_1 \) – neighbouring fluid elements A,B do not meet again at the trailing edge; C is shifted forward relative to D. To understand that, first, note that the velocity difference is proportional to the wing thickness \( h \), or in other words, to the small parameter \( h/l \). However, the difference in the path lengths can be estimated as \( l_2 - l_1 \approx h^2/l \). Therefore, for a slender wing, the upper fluid element reaches the trailing edge before the lower one. Non-zero circulation around the body in translational motion requires a wake. For a slender wing with a sharp trailing edge, the wake is very thin, like a cut, and a non-zero circulation means a jump of the potential \( \phi \) across the wake. One can generalize the method of complex potential from Sect. 1.2.4 for describing flows with circulation, which involves logarithmic terms. 26

Note that to have lift, one needs to break the up–down symmetry. Momentum conservation suggests that one can also relate the lift to the downward deflection of the flow by the body. One can also look at the pressure, which decreases towards the center of curvature of streamlines. If there is a body, wing or sail, which curves the streamlines, then the pressure decreases away from its inner part towards the center of curvature and increases away from its outer part; since the pressure far away from the body is the same on either side, then the pressure is lower on the outer side and the force is directed away from the center of curvature.

One can have a non-zero circulation and deflection of a flow without any wake simply by rotating the moving body. Since there is a non-zero circulation, then there is a deflecting (Magnus) force acting on a rotating moving sphere (Figure 1.19). That force is well known to all ball players, from soccer to tennis. The air travels faster relative to the centre of the ball where the ball surface is moving in the same direction as the air. This reduces the pressure, while on the other side of the ball the pressure increases. The result is a lift force, perpendicular to the motion. (As J.J. Thomson put it, ‘The ball follows its nose’.) One can roughly estimate the magnitude of the Magnus force from the pressure difference between the two sides, which is proportional to the
Newton argued that a rotating ball curves because the side that moves faster meets more resistance. Since he considered the resistance force proportional to the velocity squared that is to the pressure, this gives the same estimate (1.70).

The Magnus force is exploited by winged seeds, which travel away from the parent tree superimposing rotation on their descent;\textsuperscript{27} it also acts on quantum vortices moving in superfluids or superconductors (see Exercise 1.11).

Note that the drag force changes sign upon time reversal, when $v \to -v$, while the lift force and the Magnus force don’t.

Moral: wake existence teaches us that small viscosity changes the flow not only in the boundary layer but also in the whole space, both inside and outside the wake. Physically, this is because vorticity is produced in the boundary layer and is transported outside into the wake. Vorticity diffuses in low-Reynolds flows and concentrate in wakes at high Reynolds number. On the other hand, even for a very large viscosity, inertia dominates sufficiently far from the body.\textsuperscript{28}

It is instructive to think about similarities and differences in the ways that vorticity penetrating the bulk makes life interesting in classical fluids versus quantum fluids and superconductors. An evident difference is that vorticity is continuous in a classical fluid while vortices are quantized in quantum fluids. The quant of circulation $\hbar / m$ has the dimensionality of viscosity. It is also instructive to compare the Navier-Stokes equation $v_t = \nu \Delta v + \ldots$ with the Schrödinger equation, $i\hbar \psi_t = \nabla h/m) \nabla \psi + \ldots$, where $i$ makes the Laplacian term non-dissipative. Similarity is that both $\nu$ and $h/m$ are singular perturbations that introduce the highest spatial derivative and change the boundary conditions, leaving anomalies when they go to $+0$. 

\begin{align}
\Delta p \simeq \rho (u + \Omega R)^2 - (u - \Omega R)^2 / 2 = 2\rho u \Omega R.
\end{align}
1.5 Stokes flow and the wake

Exercises

1. Proceeding from the fact that the force exerted across any plane surface is wholly normal, prove that its intensity (per unit area) is the same for all aspects of the plane (Pascal’s law).

2. Consider a self-gravitating fluid with the gravitational potential $\phi$ related to the density by

$$\Delta \phi = 4\pi G \rho,$$

$G$ being the constant of gravitation. Assume spherical symmetry and static equilibrium. Describe the radial distribution of pressure for an incompressible liquid and an isothermal ideal gas.

3. Find the discharge rate from a small orifice with a cylindrical tube, projecting inward (Figure 1.20). Assume $h, S$ and the acceleration due to gravity, $g$ given. Does such a hole correspond to the limiting (smallest or largest) value of the ‘coefficient of contraction’ $S'/S$? Here $S$ is the orifice area and $S'$ is the area of the jet where contraction ceases (vena contracta).

4. Prove that if you put a little solid particle – not an infinitesimal point – at any place in the liquid it will rotate with angular velocity $\Omega$ equal to half of the local vorticity $\omega = \text{curl} \vec{v}$: $\Omega = \omega/2$.

5. There is a permanent source of water at the bottom of a large reservoir. Find the maximal elevation of the water surface for two cases:
   (i) a straight narrow slit with the constant influx $q$ (g cm$^{-1}$ s$^{-1}$) per unit length;
   (ii) a point-like source with the influx $Q$ (g s$^{-1}$).

   The fluid density is $\rho$, the depth of the fluid far away from the source is $h$. The acceleration due to gravity is $g$. Assume that the flow is potential.

6. Sketch streamlines for the potential inviscid flow and for the viscous Stokes flow in two reference systems, in which: (i) the fluid at infinity is at rest; (ii) the sphere is at rest.

7. A heavy ball with density $\rho_0$ is connected to a spring and has oscillation frequency $\omega_a$. The same ball attached to a rope makes a pendulum with oscillation frequency $\omega_b$. How do those frequencies change if such oscillators are placed in an ideal fluid with density $\rho$? What change is brought about by a small viscosity of the fluid ($\nu \ll \omega_a, b a^2$ where $a$ is the ball radius and $\nu$ is the kinematic viscosity)?

8. An underwater explosion released energy $E$ and produced a gas bubble oscillating with period $T$, which is known to be completely determined.
by $E$, the static pressure $p$ in the water and the water density $\rho$. Find the form of the dependence $T(E, p, \rho)$ up to a numerical factor. If the initial radius $a$ is known instead of $E$, can we determine the form of the dependence $T(a, p, \rho)$?

1.9 At $t = 0$ a straight vortex line exists in a viscous fluid. In cylindrical coordinates, it is described as follows: $v_r = v_z = 0$, $v_\theta = \Gamma/2\pi r$, where $\Gamma$ is some constant. Find the vorticity $\omega(r, t)$ as a function of time and the time behaviour of the total vorticity $\int \omega(r) r \, dr$.

1.10 To appreciate how one swims in a syrup, consider the so-called Purcell swimmer shown in Figure 1.21. It can change its shape by changing separately the angles between the middle link and the arms. Assume that the angle $\theta$ is small. The numbers correspond to consecutive shapes. In position 5 the swimmer has the same shape as in position 1 but moved in space. Which direction? What distinguishes this direction? How does the displacement depend on $\theta$?

1.11 In making a free kick, a good soccer player is able to utilize the Magnus force to send the ball around the wall of defenders. Neglecting vertical motion, estimate the horizontal deflection of the ball (with radius $R = 11$ cm and weight $m = 450$ g, according to FIFA rules) sent with side spin 10 revolutions per second and speed $v_0 = 30$ m s$^{-1}$ towards the goal, which is $L = 30$ m away. Take air density $\rho$ to be $10^{-3}$ g cm$^{-3}$.

1.12 Like flying, sailing also utilizes the lift (perpendicular) force acting on the sails and the keel. The fact that the wind provides a force perpendicular to the sail allows one even to move against the wind. But most optimal for starting and reaching maximal speed, as all windsurfers know, is to orient the board perpendicular to the wind and set the sail at about 45 degrees, as in Figure ?? . Why? Draw the forces acting on the board. Does the board move exactly in the direction at which the keel is pointed? Can one move faster than the wind?

1.13 Consider spherical water droplet falling under gravity.

(i) Find the fall velocity in air. Droplet radius is 0.01 mm. Air and water viscosities and densities are, respectively, $\eta_a = 1.8 \cdot 10^{-4}$ g s$^{-1}$ cm$^{-1}$, $\eta_w = 0.01$ g s$^{-1}$ cm$^{-1}$ and $\rho_a = 1.2 \cdot 10^{-3}$ g cm$^{-3}$, $\rho_w = 1$ g cm$^{-3}$; (ii) Describe the motion of an initially small droplet falling in a saturated cloud and absorbing the vapour in a swept volume so that its volume grows proportionally to its velocity and its cross-section. Consider a quasi-steady approximation, when the droplet acceleration is much less than the acceleration due to gravity, $g$. 

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FALKOVICH: “CHAP01R” — 2017/7/27 — 12:07 — PAGE 64 — #64
1.14 Consider planar free jets in an ideal fluid in the geometry shown in the figure. Find how the widths of the outgoing jets depend on the angle $2\theta_0$ between the impinging jets.

1.15 Can viscosity stop fluid rotation?

1.16 Consider spherically symmetric flow with radial velocity $v_r \propto r^{1-d}$ where $d$ is space dimensionality. Compute the vorticity, the viscous stress tensor and the viscous force. Is the flow dissipative? Does it need a pressure gradient to sustain it?

1.17 Gas flows steadily through a long pipe under the action of the pressure gradient. The pipe walls are rough so that molecules are reflected from the walls isotropically forgetting the incoming velocity. Does the mass flow rate increase or decrease with the gas density at fixed temperature? Hint: consider limits of low/high density when the pipe radius is smaller/larger than the mean free path of collisions between molecules.

1.18 An underground charge explodes at the depth $h$ far exceeding the charge size $r_0$. Most soils consist of densely packed rigid (quartz) grains which are weakly cemented. Assume that the explosion produces pressure large enough to break tangential stresses between grains and make soil flowing. When tangential stresses can be neglected, pressure-dominated flow can be considered ideal and potential. On the other hand, assume the pressure small enough so it cannot deform the grains and the flow can be considered incompressible. That usually works for pressures between hundreds and tens of thousands of bars (1 bar = $10^5$ Nm$^{-2}$). The region of crushed material is bounded by the surface at which the material velocity becomes equal some critical velocity $c$. Find the radius of the crater if the energy impacted into the fluid motion is $E$ and the soil density is $\rho$. 