2

Unsteady flows

Fluid flows can be kept steady only for very low Reynolds numbers and for velocities much less than the velocity of sound. Otherwise, either flow experiences instability and becomes turbulent or sound and shock waves are excited. Both sets of phenomena are described in this chapter.

A formal reason for instability is non-linearity of the equations of fluid mechanics. For incompressible flows, the only non-linearity is due to fluid inertia. We shall see how a perturbation of a steady flow can grow due to inertia, thus causing an instability. For large Reynolds numbers, the development of instabilities leads to a strongly fluctuating state of turbulence.

An account of compressibility, on the other hand, leads to another type of unsteady phenomena: sound waves. When density perturbation is small, velocity perturbation is much less than the speed of sound and the waves can be treated within the framework of linear acoustics. We first consider linear acoustics and discover what phenomena appear as long as one accounts for a finiteness of the speed of sound. We then consider non-linear acoustic phenomena, the creation of shocks and acoustic turbulence.

2.1 Instabilities

At large $Re$ most of the steady solutions of the Navier–Stokes equation are unstable and generate an unsteady flow called turbulence.

2.1.1 Kelvin–Helmholtz instability

Apart from a uniform flow in the whole space, the simplest steady flow of an ideal fluid is a uniform flow in a semi-infinite domain with the velocity parallel to the boundary. Physically, it corresponds to one fluid layer sliding
2.1 Instabilities

along another. Mathematically, it is a tangential velocity discontinuity, which is a formal steady solution of the Euler equation. It is a crude approximation of the description of wakes and shear flows. This simple solution is unstable with respect to arguably the simplest instability, as described by Helmholtz (1868) and Kelvin (1871). The dynamics of the Kelvin–Helmholtz instability is easy to see from Figure 2.1 where + and − denote, respectively, increase and decrease in velocity and pressure brought by surface modulation. The part of the boundary that moved towards the still region has the velocity lower, and the pressure higher, than over the part that moved towards the flow. Such pressure distribution further increases the modulation of the surface.

The perturbations \( v' \) and \( p' \) satisfy the following system of equations

\[
\text{div } v' = 0, \quad \frac{\partial v'}{\partial t} + v \frac{\partial v'}{\partial x} = - \frac{\nabla p'}{\rho}.
\]

Applying the divergence operator to the second equation we get \( \Delta p' = 0 \). This means that the elementary perturbations have the following form

\[
p_1' = \exp(i(kx - \Omega t) - kz),
\]

\[
v_1'^z = -ikp_1'/\rho_1(kv - \Omega).
\]

Indeed, the solutions of the Laplace equation that are periodic in one direction must be exponential in another direction.

The solution on the other side is obtained by setting \( v = 0 \) and \( z \to -z \). Eigenvalue \( \Omega \) is to be found from matching the solutions at the boundary. To relate the upper side (indexed 1) to the lower side (indexed 2) we introduce \( \zeta(x,t) \), the elevation of the surface, its time derivative is the \( z \) component of the velocity:

\[
\frac{d\zeta}{dt} = \frac{\partial \zeta}{\partial t} + v \frac{\partial \zeta}{\partial x} = v_1^z,
\]

(2.1)

that is \( v_1^z = i\zeta(kv - \Omega) \) and \( p_1' = -\zeta\rho_1(kv - \Omega)^2/k \). On the other side, we can express in a similar way \( p_2' = \zeta\rho_2\Omega^2/k \). The pressure is continuous across the
surface, which gives the matching condition:

$$\rho_1 (k v - \Omega)^2 = -\rho_2 \Omega^2 \Rightarrow \Omega = k v \frac{\rho_1 \pm i \sqrt{\rho_1 \rho_2}}{\rho_1 + \rho_2}. \tag{2.2}$$

Positive $\text{Im} \Omega$ means an exponential growth of perturbations, i.e. instability.\(^1\)

The largest growth rate corresponds to the largest admissible wavenumber. In reality the layer, where velocity increases from zero to $v$, has some finite thickness $\delta$ and our approach is valid only for $k\delta \ll 1$.

A complementary insight into the physics of the Kelvin–Helmholtz instability can be obtained by considering vorticity. In the unperturbed flow, the vorticity $\partial v_x / \partial z$ is concentrated in the transitional layer, which is thus called the vortex layer (or vortex sheet when $\delta \to 0$). One can consider a discrete version of the vortex layer as a chain of identical vortices, shown in Figure 2.2. Owing to symmetry, such an infinite array of vortex lines is stationary since the velocities imparted to any given vortex by all the others cancel. The small displacements shown by straight arrows in Figure 2.2 lead to an instability with the vortex chain breaking into pairs of vortices circling round one another. That circling motion turns an initially sinusoidal perturbation into spiral rolls during the non-linear stage of the evolution, as shown in Figure 2.3, obtained experimentally. The Kelvin–Helmholtz instability in the atmosphere is often made visible by corrugated cloud patterns, as seen in Figure 2.4; similar patterns are seen on sand dunes. It is also believed to be partially responsible for

![Figure 2.2](image1.png)

Figure 2.2 The array of vortex lines is unstable with respect to the displacements, shown by straight arrows.

![Figure 2.3](image2.png)

Figure 2.3 Spiral vortices generated by the Kelvin–Helmholtz instability. Photograph by F. Roberts, P. Dimotakis and A. Rosbko, reproduced from [27].
clear air turbulence (that is atmospheric turbulence unrelated to moist convection). Numerous manifestations of this instability are found in astrophysics, from the interface between the solar wind and the Earth’s magnetosphere to the boundaries of galactic jets.

The vortex view of the Kelvin–Helmholtz instability suggests that a unidirectional flow depending on a single transverse coordinate, like \( v_x(z) \), can only be unstable if it has a vorticity maximum on some surface. Such a vorticity maximum is an inflection point of the velocity since \( \frac{d\omega}{dx} = \frac{d^2v_x}{dz^2} \). This explains why flows without inflection points are linearly stable (Rayleigh, 1880). Examples of such flows are plane linear profiles, flows in a pipe or between two planes driven by pressure gradients, flows between two planes moving with different velocities, etc.\(^2\) In particular, the vortex layer can be locally considered as a linear profile for perturbations with \( k\delta \gg 1 \), so such perturbations cannot grow. Therefore, the maximal growth rate corresponds to \( k\delta \simeq 1 \), i.e. the wavelength of the most unstable perturbation is comparable to the layer thickness.
Our consideration of the Kelvin–Helmholtz instability was for completely inviscid fluids which presumes that the effective Reynolds number was large: $Re = v\delta/\nu \gg 1$. In the opposite limit when the friction is very strong, the velocity profile is not stationary but rather evolves according to the equation $\partial v_z(z,t)/\partial t = v\partial^2 v_z(z,t)/\partial z^2$, which describes the thickness growing as $\delta \propto \sqrt{vt}$. Such a diffusing vortex layer is stable because the friction damps all the perturbations. It is thus clear that there must exist a threshold Reynolds number above which instability is possible. We now consider this threshold from a general energetic perspective.
2.1 Instabilities

2.1.2 Energetic estimate of the stability threshold

The energy balance between the unperturbed steady flow \( v_0(r) \) and the superimposed perturbation \( v_1(r, t) \) helps one to understand the role of viscosity in imposing an instability threshold. Consider the flow \( v_0(r) \), which is a steady solution of the Navier–Stokes equation:

\[
(\nabla p_0/\rho + \nu_1 \nabla v_0) = (\nabla \cdot v_0) v_0.
\]

The perturbed flow \( v_0(r) + v_1(r, t) \) satisfies the equation:

\[
\frac{\partial v_1}{\partial t} + (v_1 \cdot \nabla) v_0 + (v_0 \cdot \nabla) v_1 + (v_1 \cdot \nabla) v_1 = -\nabla p_1/\rho + \nu \Delta v_1.
\] (2.3)

Making the scalar product of (2.3) with \( v_1 \) and using incompressibility one gets:

\[
\frac{1}{2} \frac{\partial v_1^2}{\partial t} = -v_1 v_k \frac{\partial v_{1i}}{\partial x_k} \frac{1}{Re} \frac{\partial v_{1i}}{\partial x_k}
\]

\[
-\frac{\partial}{\partial x_k} \left[ \frac{v_1^2}{2} (v_{0k} + v_{1k}) + p_1 v_{1k} - \frac{v_{1i} \frac{\partial v_{1i}}{\partial x_k}}{Re} \right].
\]

The last term disappears after integration over the volume:

\[
\frac{d}{dt} \int \frac{v_1^2}{2} d\mathbf{r} = T - \frac{D}{Re},
\] (2.4)

\[
T = -\int v_i v_{1k} \frac{\partial v_{1i}}{\partial x_k} d\mathbf{r},
\]

\[
D = \int \left( \frac{\partial v_{1i}}{\partial x_k} \right)^2 d\mathbf{r}.
\]
2 Unsteady flows

The term $T$ is due to inertial forces and the term $D$ is due to viscous friction. We see that for stability (i.e., for decay of the energy of the perturbation) one needs friction to dominate over inertia (Reynolds 1894):

$$Re < Re_E = \min_{v_1} \frac{D}{T}. \quad (2.5)$$

The minimum is taken over different perturbation flows. Since both $T$ and $D$ are quadratic in the perturbation velocity, their ratio depends on the orientation and spatial dependence of $v_1(r)$ but not on its magnitude. Since a uniform flow is stable, then it is natural that one needs $\partial v_0/\partial r \neq 0$ to provide non-zero energy input $T$. Moreover, one must have the perturbation velocity oriented in such a way as to have both the component $v_{1i}$ along the mean flow and the component $v_{1k}$ along the gradient of the mean flow. Then $v_{1i}v_{1k}$ is the flux of the momentum along the flow in the direction of the flow gradient. To have positive $T$ one needs this flux to have the sign opposite to the flow gradient — in this case, the perturbation diminishes the flow gradient and the flow energy, thus increasing the perturbation energy. An example of such a geometry is shown in Figure 2.5. While the flow is always stable for $Re < Re_E$, it is not necessary unstable when one can find a perturbation that breaks (2.5); for instability to develop, the perturbation must also evolve in such a way as to keep $T > D$. As a consequence, the critical Reynolds numbers are usually somewhat higher than those given by the energetic estimate.

2.1.3 Landau’s law

Dimensionless parameter, such as Reynolds number, whose change can bring qualitative changes in behavior, is called control parameter. When the control parameter passes a critical value, the system undergoes an instability and goes into a new state. Generally, one cannot say much about this new state
2.1 Instabilities

except for the case when it is not very much different from the old one. That may happen when the control parameter is not far from critical. Consider $Re > Re_{cr}$ but $Re - Re_{cr} \ll Re_{cr}$. Just above the instability threshold, there is usually only one unstable mode. Let us linearize the equation (2.3) with respect to the perturbation $v_1(\mathbf{r}, t)$, i.e. omit the term $(v_1 \cdot \nabla)v_1$. The resulting linear differential equation with time-independent coefficients has a solution of the form $v_1 = f_1(\mathbf{r}) \exp(\gamma_1 t - i\omega_1 t)$ which describes the unstable mode. The exponential growth has to be restricted by the terms that are non-linear in $v_1$. The solution of a weakly non-linear equation can be sought in the form $v_1 = f_1(\mathbf{r}) A(t)$. The equation for the amplitude $A(t)$ must generally have the following form: $\frac{d|A|^2}{dt} = 2\gamma_1 |A|^2 - \alpha |A|^4 + \text{third-order terms + \ldots}$. The fourth-order terms are obtained by expanding $v = v_0 + v_1 + v_2$ further and accounting for $v_2 \propto v_1^2$ in the equation for $v_1$. The growth rate turns into zero at $Re = Re_{cr}$ and generally $\gamma_1 \propto Re - Re_{cr}$, while the frequency is usually finite at $Re \to Re_{cr}$. We can thus average the amplitude equation over time larger than $2\pi/\omega_1$ but smaller than $1/\gamma_1$. Since the time of averaging contains many periods, then among the terms of the third and fourth order only $|A|^4$ gives a non-zero contribution:

$$\frac{d|A|^2}{dt} = 2\gamma_1 |A|^2 - \alpha |A|^4. \quad (2.6)$$

Since the averaging time is much less than the time of the modulus change, then one can remove the overbar in the left-hand side of (2.6) and solve it as a usual ordinary differential equation. This equation has the solution

$$|A|^{-2} = \alpha/2\gamma_1 + \text{const} \cdot \exp(-2\gamma_1 t) \to \alpha/2\gamma_1.$$

The saturated value changes with the control parameter according to the so-called Landau’s law:

$$|A|_{\text{max}}^2 = \frac{2\gamma_1}{\alpha} \propto Re - Re_{cr}.$$

We thus see that nonlinearity (i.e. inertia) has a dual role: it overcomes friction to make the instability possible but then it stops the growth of the perturbation at a finite amplitude. If $\alpha < 0$ then one needs a $-\beta |A|^6$ term in (2.6) to stabilize the instability.

$$\frac{d|A|^2}{dt} = 2\gamma_1 |A|^2 - \alpha |A|^4 - \beta |A|^6. \quad (2.7)$$

The saturated value is now

$$|A|_{\text{max}}^2 = -\frac{\alpha}{2\beta} \pm \sqrt{\frac{\alpha^2}{4\beta^2} + 2\gamma_1/\beta}.$$
Stability with respect to the variation of $|A|^2$ within the framework of (2.7) is determined by the factor $2\gamma_1 - 2\alpha |A|^2_{\text{max}} - 3\beta |A|^4_{\text{max}}$. Between B and C, the steady flow is metastable. The broken curve is unstable.

This description is based on the assumption that at $Re - Re_c \ll Re_c$ the only important dependence is $\gamma_1(Re)$ very much like in Landau’s theory of phase transitions (which also treats loss of stability). The amplitude $A$, which is non-zero on one side of the transition, is an analogue of the order parameter. Cases of positive and negative $\alpha$ correspond to the phase transitions of the second and first order, respectively.

### 2.2 Turbulence

As the Reynolds number increases beyond the threshold of the first instability, it eventually reaches a value where the new periodic flow becomes unstable in its own turn with respect to another type of perturbation, usually with smaller scale and consequently higher frequency. Every new instability brings about an extra degree of freedom, characterized by the amplitude and the phase of the new periodic motion. The phases are determined by (usually uncontrolled) initial perturbations. At very large $Re$, a sequence of instabilities produces turbulence as a superposition of motions of different scales (Figure 2.6). The resulting flow is irregular both spatially and temporally so we need to describe it statistically.

Flows that undergo instabilities usually become temporally chaotic at moderate $Re$ because motion in the phase space of more than three interacting degrees of freedom may tend asymptotically to sets (called attractors), which are more complicated than fixed points (steady states), cycles (periodic motions) or tori (multi-periodic motions). Namely, there exist attractors, called strange or chaotic, that consist of saddle-point trajectories. Such trajectories have stable directions, by which the system approaches the attractor, and unstable directions, lying within the attractor. Because all trajectories are unstable on the attractor, any two initially close trajectories separate exponentially at a mean rate called the Lyapunov exponent. To intuitively appreciate how the mean stretching rate can be positive in a random flow, note that around a saddle-point
2.2 Turbulence

more vectors undergo stretching than contraction (Exercise 2.1). Exponential separation of trajectories means instability and unpredictability of the flow patterns. The resulting fluid flow that corresponds to a strange attractor is regular in space and random in time; it is called dynamical chaos.\(^3\) One can estimate the Lyapunov exponent for the Earth’s atmosphere by dividing the typical wind velocity \(20 \text{ m s}^{-1}\) by the global scale 10000 km. The inverse Lyapunov exponent gives the time one can reasonably hope to predict weather, which is \(10^7 \text{ m}/(20 \text{ m s}^{-1}) = 5 \cdot 10^5 \text{ s},\) i.e. about a week.

The laminar flow can be linearly stable at large \(Re\) (as uni-directional flows without inflection points) or at all \(Re\) (as pipe flows). However, the basin of attraction of such a flow shrinks when \(Re\) grows, so that fluctuations of small yet finite amplitude are able to excite turbulence, which then sustains itself. Turbulence onset is of probabilistic nature in this case: for any finite \(Re\), there is a finite probability for the flow to return to a laminar state. For example, some fraction of the pipe flow is always laminar. This is because there is always a finite probability for any given perturbation either to decay or to expand/split. The mean decay time increases while the mean split time decreases with \(Re\); one can define the critical \(Re\) as when the times are equal.\(^4\) As the Reynolds number increases, expansion and splitting of perturbations leads to filling the space with turbulence. On the contrary, linearly unstable laminar flow cannot exist beyond the critical value of the Reynolds number or other control parameter. In such systems, turbulence arises from an increase in temporal complexity of fluid motion.

Figure 2.6 Instabilities in three almost identical convective jets lead to completely different flow patterns. Notice also the appearance of progressively smaller scales as the instabilities develop. Photograph copyright: Vbotond, www.dreamstime.com.
2.2.1 Cascade

Here we consider turbulence at very large $Re$. Turbulence is a flow that is random in space and in time. Such flows require a statistical description that is able to predict mean (expectation) values of different quantities. Despite five centuries of effort (since Leonardo da Vinci) a complete description is still lacking but some important elements have been established. We do not yet know exactly which properties of developed turbulence are independent of the path taken to $Re \to \infty$, in particular, whether it started from a linearly unstable laminar flow and went through stochastic attractors or needed finite perturbations of a linearly stable flow. However, a revealing insight into the universal aspects of turbulence is given a cascade picture (Figure 2.7), which I present in this section. It is a useful phenomenology, both from a fundamental viewpoint of understanding a state with many degrees of freedom deviated from equilibrium and from a practical viewpoint of explaining the empirical fact that the drag force is finite in the inviscid limit. The finiteness of the drag coefficient,
2.2 Turbulence

\[ C(Re) = \frac{F}{\rho u^3 L^2} \rightarrow \text{const. at } Re \rightarrow \infty \] (see Figure ??), means that the rate of the kinetic energy input per unit mass, \( \epsilon = \frac{F}{\rho u L^3} = C u^3 / 2L \), stays finite when \( \nu \rightarrow 0 \). Where does all this energy go if we consider not an infinite wake but a bounded flow, say, generated by a permanently acting fan in a room? Experiment (and everyday experience) tells us that a fan generates some air flow whose magnitude stabilizes after a while, which means that the input is balanced by the viscous dissipation. Since the input is expected to be independent of small viscosity, this means that the energy dissipation rate \( \epsilon = \nu \int \omega^2 \, dV / V \) stays finite when \( \nu \rightarrow 0 \) (if the fluid temperature is kept constant).

Historically, the understanding of turbulence as cascade started from an empirical law established by Richardson (observing seeds and balloons released in the wind): the mean squared distance between two particles in turbulence increases in a super-diffusive way, \( \langle R^2(t) \rangle \propto t^3 \). Here the average is over different pairs of particles. The parameter that can relate \( \langle R^2(t) \rangle \) and \( t^3 \) must have dimensionality cm\(^2\) s\(^{-3}\), which is that of the dissipation rate \( \epsilon \): \( \langle R^2(t) \rangle \simeq \epsilon t^3 \).

Richardson’s law can be interpreted as the increase of the typical velocity difference \( \delta v(R) \) with distance \( R \): since there are vortices of different scales in a turbulent flow, the velocity difference at a given distance is due to vortices with comparable scales and smaller; as the distance increases, more (and larger) vortices contribute to the relative velocity, which makes separation faster than diffusive (when the velocity is independent of the distance). Richardson’s law suggests the law of the relative velocity increase with the distance in turbulence. Indeed, \( R(t) \simeq e^{1/3} t^{1/3} \) is a solution of the equation \( dR / dt = (\epsilon R)^{1/3} \); since \( dR / dt = \delta v(R) \) then

\[ \delta v(R) \simeq (\epsilon R)^{1/3} \Rightarrow \frac{(\delta v)^3}{R} \simeq \epsilon. \] (2.8)

The last relation brings the idea of the energy cascade over scales, which goes from the scale \( L \) with \( \delta v(L) \simeq u \) down to the viscous scale \( l \) defined by \( l \delta v(l) \simeq \nu \).

The viscous scale is much larger than the mean free path as long as \( \delta v(L) \ll c \).

The energy flux through the given scale \( R \) can be estimated as the energy \( (\delta v)^2 \) divided by the time \( R / \delta v \). For the so-called inertial interval of scales, \( L \gg R \gg l \), there is neither force nor dissipation so that the energy flux \( \epsilon(R) = \langle \delta v^3(R) \rangle / R \) may be expected to be \( R \)-independent, as suggested by (2.8). When \( \nu \rightarrow 0 \), the viscous scale \( l \) decreases, that is the cascade gets longer, but the amount of the flux and the dissipation rate stay the same. In other words, finiteness of \( \epsilon \) in the limit of vanishing viscosity can be interpreted as locality of the energy transfer in \( R \)-space (or equivalently, in Fourier space). By using an analogy, one may say that turbulence is supposed to work as a pipe with a flux through its cross-section independent of the length of the pipe.\(^5\)
Let us illustrate the cascade picture by an estimate. For a wind speed $v \approx 10 \text{ m/sec}$ at the height $L \approx 150 \text{ m}$, we shall have $Re \approx 10^8$ and the viscous scale can be estimated from $l \approx l v(L)/L \approx v$, which gives $l = L Re^{-3/4} \approx 0.15 \text{ mm}$. So wind turbulence contains vortices from hundreds of meters to the fraction of a millimeter.

The cascade picture is a nice phenomenology but can one support it with any derivation? One can obtain an exact relation that quantifies the flux constancy (Kolmogorov 1941). Let us derive first the equation for the correlation function of the velocity at different points for an idealized turbulence whose statistics are presumed isotropic and homogeneous in space. We assume no external forces so that the turbulence must decay with time. Let us find the time derivative of the correlation function of the components of the velocity difference between the points 1 and 2,

$$\langle (v_1 - v_2)(v_{1k} - v_{2k}) \rangle = \frac{2\langle v^2 \rangle}{3} \delta_{ik} - 2\langle v_2, v_{1k} \rangle.$$  

The time derivative of the kinetic energy is minus the dissipation rate: $\epsilon = -\frac{d}{dt} \langle v_r^2 \rangle / 2$. To obtain the time derivative of the two-point velocity correlation function, take the Navier–Stokes equation at some point $r_1$, multiply it by the velocity $v_2$ at another point $r_2$ and average it over time intervals larger than $|r_1 - r_2|/|v_1 - v_2|$ and smaller than $L/u$:

$$\partial_t \langle v_1, v_{2k} \rangle = -\partial_{x_1} \langle v_{1i} v_{1i}, v_{2k} \rangle - \frac{1}{\rho} \partial_{x_1} \langle p_1, v_{2k} \rangle - \frac{1}{\rho} \partial_{x_2} \langle p_2, v_{1i} \rangle + \nu (\Delta_1 + \Delta_2) \langle v_{1i} v_{2k} \rangle.$$  

It is presumed that the temporal average is equivalent to the spatial average, property called ergodicity. Statistical isotropy means that the vector $\langle p_1, v_2 \rangle$ has nowhere to look but to $r = r_1 - r_2$, the only divergence-less such vector, $r/r^3$, does not satisfy the finiteness at $r = 0$ so that $\langle p_1, v_2 \rangle = 0$. Owing to the space homogeneity, all the correlation functions depend only on $r = r_1 - r_2$:

$$\partial_t \langle v_1, v_{2k} \rangle = -\partial_{x_1} \left( \langle v_{1i} v_{1i}, v_{2k} \rangle + \langle v_{2i}, v_{1k} \rangle \right) + 2\nu \Delta \langle v_{1i} v_{2k} \rangle.$$  

We have used here $\langle v_{1i} v_{2k} v_{2j} \rangle = -\langle v_{2i} v_{1k} v_{1j} \rangle$ since under $1 \leftrightarrow 2$ both $r$ and a third-rank tensor change sign (the tensor turns into zero when $1 \to 2$). By straightforward yet lengthy derivation one can rewrite (2.9) for the moments of the longitudinal velocity difference, called structure functions,

$$S_n(r, t) = \langle r \cdot (v_1 - v_2)^n \rangle / r^n.$$
2.2 Turbulence

This gives the so-called Kármán–Howarth relation

$$\frac{\partial S_2}{\partial t} = -\frac{1}{3} r^4 S_3 - \frac{4\epsilon}{3} r^2 S_2 \frac{\partial S_2}{\partial r} + \frac{2\nu}{r^2} r^4 \frac{\partial S_2}{\partial r}.$$  \hspace{1cm} (2.10)

The average quantity $S_2$ changes only together with a large-scale motion, so

$$\frac{\partial S_2}{\partial t} \sim S_2 u \ll S_3 \quad \text{at } r \ll L.$$  \hspace{1cm} (2.11)

That remarkable relation tells that turbulence is irreversible since $S_3$ does not change sign when $t \to -t$ which requires $v \to -v$. If screen a movie of turbulence backwards, we can indeed tell that something is wrong! This is what is called ‘anomaly’ in modern field-theoretical language: a symmetry of the inviscid equation (here, time-reversal invariance) is broken by the viscous term even though the latter might have been expected to become negligible in the limit $\nu \to 0$. In other words, the effect of symmetry breaking remains finite when symmetry-breaking factor goes to zero.

It is instructive to recall that the statistics is reversible in thermal equilibrium: the detailed balance principle states that the probabilities of every process and its time reversal are equal. This is related to the fact that the thermostat provides both dissipation and short-correlated forcing, which balance each other at every scale and timescale, as expressed by the fluctuation-dissipation theorem. On the contrary, irreversibility of turbulence statistics can be traced to the fact that forcing and dissipation act on different scales.

Here the good news ends. There is no analytic theory to give us other structure functions. One may assume following Kolmogorov (1941) that $\epsilon$ is the only quantity determining the statistics in the inertial interval, then on dimensional grounds $S_\alpha \approx (\epsilon r)^n$. Experiment gives the power laws, $S_\alpha(r) \propto r^n$ but with the exponents $\zeta_n$ deviating from $n/3$ for $n \neq 3$. Moments of the velocity difference can be obtained from the probability density function (PDF), which describes the probability of measuring the velocity difference $\delta v = u$ at distance $r$: $S_\alpha(r) = \int u^n P(u, r) du$. Deviations of $\zeta_n$ from $n/3$ mean that the PDF $P(\delta v, r)$ is not scale invariant, i.e. cannot be presented as $(\delta v)^{-1} \times \text{dimensionless function of single variable } \delta v / (\epsilon r)^{1/3}$. Apparently, there is more to turbulence than just a cascade, and $\epsilon$ is not all one must know to predict the statistics of the velocity. Similar breakdown of scale invariance takes place in the
simpler one-dimensional case of Burgers turbulence described in Section 2.3.4 below, where it can be related to shock waves. One can also relate breakdown of scale invariance in turbulence to statistical integrals of motion of fluid particles. For example, for two particles with the coordinates \( R_1(t) \) and \( R_2(t) \) and velocities \( v_1(t) \) and \( v_2(t) \) the quantity 
\[
\langle |v_1 - v_2|^2 |R_1 - R_2|^2 \rangle
\]
does not change at \( t \to \infty \). Both symmetries, one broken by pumping (scale invariance) and another by friction (time reversibility) are not restored even when \( r/L \to 0 \) and \( l/r \to 0 \).

To appreciate difficulties in turbulence theory, one can cast the turbulence problem into that of quantum field theory. Consider the Navier–Stokes equation driven by a random force \( f \) with the Gaussian probability distribution \( P(f) \) defined by the variance 
\[
\langle f_i(0,0) f_j(r,t) \rangle = D_{ij}(r,t).
\]
Then the probability of any flow \( v(r,t) \) is given by the Feynman path integral over velocities satisfying the Navier–Stokes equation with different force histories:
\[
\int Dv Df \delta \left( \partial_t v + (v \cdot \nabla) v + \nabla P/\rho - v \Delta v - f \right) P(f) = \int Dv Dp \exp \left[ -D_{ij} p_i p_j + i p_i (\partial_t v_i + v_k \nabla_k v_i + \nabla_i P/\rho - v \Delta v_i) \right].
\] (2.12)

Here we have presented the delta function as an integral over an extra field \( p \) and explicitly made Gaussian integration over the force. One can thus see that turbulence is equivalent to the field theory of two interacting fields \( (v \text{ and } p) \) with large \( Re \) corresponding to a strong coupling limit (for incompressible turbulence the pressure is recovered from \( \text{div}(v \cdot \nabla v) = \Delta P \)). For fans of field theory, add that the convective derivative \( d/dt = \partial/\partial t + (v \cdot \nabla) \) can be identified as a covariant derivative in the framework of a gauge theory; here the velocity of the reference frame fixes the gauge.

### 2.2.2 Turbulent flows

With the new knowledge of turbulence as a multi-scale flow, let us now return to the large-Reynolds flows down an inclined plane and past the body.

**River.** Now that we know that turbulence makes the drag at large \( Re \) much larger than the viscous drag for a laminar flow, we can understand why the behaviour of real rivers is so distinct from a laminar solution from Section ??.

Denote the mean velocity as \( U \). At small \( Re \), the momentum injected by gravity into the fluid is carried to the bottom by the viscous momentum flux, that is the gravity force (per unit mass) \( ga \) is balanced by the viscous drag \( vU/h^2 \), or equivalently the input power \( gaU \) is equal to the rate of the viscous dissipation \( vU^2/h^2 \). At large \( Re \), we expect turbulence with the energy dissipation rate and drag force independent of the viscosity. Then the power \( gaU \) must be equal to the dissipation rate which can be estimated as \( U^3/h \), and the drag \( U^2/h \)
balances $g\alpha$ so that

$$U \simeq \sqrt{ag\alpha}.$$  \hfill (2.13)

Indeed, as long as viscosity does not enter into things, this is the only combination with the velocity dimensionality that one can get from $h$ and the effective gravity $ag$. For slow plain rivers (inclination angle $\alpha \simeq 10^{-4}$ and depth $h \simeq 10\text{m}$), the new estimate (2.13) gives a reasonable $v \simeq 10\text{cm s}^{-1}$. Another way to describe the drag is to say that molecular viscosity $\nu$ is replaced by turbulent viscosity $\nu_T \simeq \nu h \simeq v Re$ and the drag is still given by the viscous formula $\nu U/2h^2$ but with $\nu \to \nu_T$. Intuitively, one imagines turbulent eddies transferring momentum between fluid layers.

Similar arguments can be applied to flows in pipes and channels under the action of the pressure gradient, replacing in (2.13) $ag$ by $\nabla P/\rho$. One is then tempted to conclude that the dimensionless friction factor, defined as $\nabla Ph/\rho U^2$ or $agh/U^2$, decreases with $Re$ as $1/Re$ at small $Re$ and saturates to a constant at large $Re$, just like the drag coefficient shown in Figure ??

A closer look reveals, however, a flaw in the argument leading to (2.13) — it treats the mean flow and turbulence as homogeneous, while they must be $z$-dependent to carry the momentum injected by gravity or pressure gradient towards the bottom or walls to be absorbed there. Let us write the momentum conservation without assuming the flow unidirectional. Denote the velocity $x$-component as $U(z) + u(x, y, z, t)$ and $z$-component as $v(x, y, z, t)$, where apparently $u, v$ describe turbulent fluctuations. Then the continuity equation

![Figure 2.8 Sketch of the wake behind a body.](image-url)
for the $x$-component of the mean momentum states that the divergence of the momentum flux $\tau$ is equal to the force:

$$
\frac{d}{dz} \left( \nu \frac{dU}{dz} + \langle uv \rangle \right) \equiv \frac{d\tau(z)}{dz} = -\alpha gh.
$$

(2.14)

Integrating we get $\tau(z) = \tau(0) - \alpha g z$. The flux is zero on the river surface and at the center of a pipe or channel which gives $\tau(0) = \alpha gh$. Let us now consider the flow at $z \ll h$ where the momentum flux can be considered independent of $z$, and denote $v^2_0 \equiv \tau(0) = \alpha gh$. In this region the mean velocity is independent of $h$ and must depend only on $v, z, v_*$. By dimensional reasoning the dependence must have a form $U = v_* f(z v_* / \nu)$. The dimensionless parameter $z v_* / \nu$ is the Reynolds number with the scale set by the distance to the solid boundary. Near the boundary, viscosity absorbs the flux: $\nu dU/dz = \alpha g$ and $U(z) = \alpha g h z / \nu$. The width of that viscous boundary layer can be estimated requiring the Reynolds number to be of order unity: $l = \nu / v_*$. Outside of this layer, for $z \gg l$, one may expect viscosity to be unimportant and the flux carried by turbulence. As we cannot yet develop a consistent theory of such inhomogeneous turbulence, let us use plausible arguments. Since there is no momentum flux in a uniform flow, then it is natural to relate the mean flux to the mean velocity gradient, $dU/dz$, which must be determined solely by $v_*$ and $z$ at $l \ll z \ll h$. The only dimensionally possible relation is $dU/dz \approx v_* / z$, which gives a logarithmic velocity profile for turbulent boundary layer (Karman 1930, Prandtl 1932):

$$
U(z) \approx v_* \log(z/l) = \sqrt{\alpha gh \log(\alpha gh)^{1/2}}.
$$

(2.15)

We used $l$ to make the argument of the logarithm dimensionless since for $z \approx l$ one must have $U(l) \approx v_*$. One can further illuminate the hypothesis underlying the log law (2.15) using so-called overlap argument. The dimensionless quantity $U(z)/v_*$ must be a function of two dimensionless arguments, $\ell = z/h$ and $Re = v_* h / \nu$. Near the wall we expect $h$ to disappear: $U(z)/v_* \to f(\ell Re)$. Near the center, we expect $v_*$ to disappear from the law of the velocity change: $U(h) - U(z) = v_0 f_1(\ell)$. Denote $U(h)/v_* = f_2(Re)$. We now make an assumption that the two asymptotic regions overlap. In this overlap region we have $f(\ell Re) = f_2(Re) - f_1(\ell)$, which requires all the functions to be logarithmic. Logarithmic turbulent profile is more flat than parabolic laminar profile, which is natural since turbulence better mixes momentum. The overlap argument and claim that the momentum flux completely determines the mean flow in a turbulent boundary layer are curiously similar to assuming inertial interval with the energy flux determining everything in the cascade picture of homogenous turbulence. We now know that Kolmogorov-Obukhov theory correctly describes only the third
moment of the velocity statistics, while other moments depend on the large scale. It is not yet clear whether the Prandtl-Karman theory must be modified in a similar way. Experiments support logarithmic mean flow profile but show that turbulence statistics depends on $h$ even at $z \ll h$.

We see that (2.15) corrects (2.13) by a viscosity-dependent logarithmic factor. That makes velocity everywhere, even outside of the viscous layer, dependent on viscosity. While this dependence is very slow and for most cases negligible, conceptually it has dramatic consequences. It tells us that when viscosity goes to zero, the width $l$ of the viscous layer shrinks to zero but $U(l) \simeq u_*$ i.e. stays finite. That means that we have an effective slip on the solid boundary. At any finite $z$, the velocity $U(z)$ goes to infinity, so that the friction factor goes to zero at $v \to 0$ as $\log^{-2}(h u_*/v)$. All this is because we consider the boundary straight and smooth, which explains the dramatic difference from the flow past a body, where curved surface provides for separation of the boundary layer and resulting wake provides for a finite drag coefficient. It is then reasonable to assume that if the logarithmic decrease of the friction factor with the Reynolds number takes place, it stops when $l$ is getting comparable to the size $r$ of the boundary inhomogeneities (which again save the day, as in Section ??, but in a different way). When $v < r u_*$ one cannot assume the mean flow to be parallel to the solid boundary. Every inhomogeneity then provides its own wake with a finite drag so that $U(r) \simeq u_*$ and the friction factor saturates at $\log^{-2}(h/r)$.

**Wake.** Let us now describe the entire wake behind a body at $Re = u L/v \gg 1$. Since $Re$ is large, Kelvin’s theorem holds outside the boundary layer – every streamline keeps its vorticity. Streamlines are thus divided into those of zero and non-zero vorticity. A separated region of rotational flow (wake) can exist only if streamlines don’t go out of it. Yet zero-vorticity streamlines may come in so that the wake grows as one goes away from the body. Velocity is lower in the wake than outside, and instability of the Kelvin–Helmholtz type makes the boundary of the wake wavy. Oscillations then must also be present in the velocity field in the immediate outside vicinity of the wake. Still, only large-scale harmonics of turbulence are present in the outside region, because the flow is potential ($\Delta \phi = 0$) so when it changes periodically along the wake it decays exponentially with distance from the wake boundary. The smaller the scale, the faster it decays away from the wake. Therefore, all the small-scale motions and all the dissipation are inside the turbulent wake. The boundary of the turbulent wake fluctuates in time. In the snapshot sketch in Figure 2.8, the wake is dark, broken lines with arrows are streamlines; see Figure ?? for a real wake photo.
Let us describe the time-averaged position of the wake boundary $Y(x)$. The average angle between the streamlines and the $x$-direction is $v(x)/u$, where $v(x)$ is the rms turbulent velocity, which can be obtained from the condition that the momentum flux through the wake must be $x$-independent since it is equal to the drag force $F \simeq \rho u v Y^2$, as in (??). Then

$$\frac{dY}{dx} = \frac{v(x)}{u} \simeq \frac{F}{\rho u^3 Y^2},$$

so that

$$Y(x) \simeq \left( \frac{Fx}{\rho u^2} \right)^{1/3}, \quad v(x) \simeq \left( \frac{Fu}{\rho x^2} \right)^{1/3}.$$

One can substitute $F \simeq \rho u^2 L^2$ and get

$$Y(x) \simeq L^{2/3} x^{1/3}, \quad v(x) \simeq u(L/x)^{2/3}.$$ 

Note that $Y$ is independent of $u$ for a turbulent wake. The current Reynolds number, $Re(x) = v(x)Y(x)/v \simeq (L/x)^{1/3} uL/v = (L/x)^{1/3} Re$, decreases with $x$ and a turbulent wake turns into a laminar one at $x > L Re^3 = L(uL/v)^3$ – the transition distance apparently depends on $u$.

Inside the laminar wake, under the assumption $v \ll u$, we can neglect $\rho^{-1} \partial p/\partial x \simeq v^2/x$ in the steady Navier–Stokes equation, which then turns into the (parabolic) diffusion equation with $x$ playing the role of time:

$$u \frac{\partial v_x}{\partial x} = \nu \left( \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} \right) v_x. \quad (2.16)$$

At $x \gg v/u$, the solution of this equation acquires the universal form

$$v_x(x, y, z) = -\frac{F_x}{4\pi \eta x} \exp \left[ -\frac{u(z^2 + y^2)}{4vx} \right],$$

Figure 2.9 Wake width $Y$ versus distance from the body $x$. 

\[ \text{turbulent} \]

\[ \text{laminar} \]

\[ x^{1/3} \]

\[ x^{2/3} \]

\[ L \]

\[ L Re^3 \]
where we have used (?) in deriving the coefficient. A prudent thing to ask now is why we accounted for the viscosity in (2.16) but not in the stress tensor (?). The answer is that \( \sigma_{xx} \propto \partial v_x/\partial x \propto 1/x^2 \) decays fast with the distance while \( \int dy \sigma_{xy} = \int dy \partial v_x/\partial y \) vanishes identically.

We see that the laminar wake width is \( Y \approx \sqrt{\nu x/u} \), that is the wake is parabolic. The Reynolds number further decreases in the wake according to the law \( v_x Y/\nu \propto x^{-1/2} \). Recall that in the Stokes flow \( v \propto 1/r \) only for \( r < \nu/u \), while in the laminar wake \( v_x \propto 1/x \) ad infinitum. Comparing laminar and turbulent estimates, we see that for \( x \ll L Re^3 \), the turbulent estimate gives a larger width: \( Y \approx L^{2/3} x^{1/3} \gg (\nu x/u)^{1/2} \). On the other hand, in a turbulent wake the width grows and the velocity perturbation decreases with distance slower than in a laminar wake (Figure 2.9).

Wake consideration can be readily generalized for an arbitrary space dimensionality \( d \), where \( F \approx \rho u v Y^{d-1} \), so that \( v \propto Y^{1-d} \) and \( Re \propto v Y \propto Y^{2-d} \). We see that wake expansion does not cause the decrease of the Reynolds number in two dimensions. For a body extended across the flow (like a long wing), one can consider the wake two-dimensional and the Reynolds number approximately constant until the distance comparable to the body span.

### 2.2.3 Mixing

The diffusivity of gases in gases is \( \kappa \approx 10^{-1} \text{cm}^2 \text{s}^{-1} \) and liquids in liquids \( \kappa \approx 10^{-5} \text{cm}^2 \text{s}^{-1} \), so it would take many hours for a smell to diffuse across the dinner table and milk across the coffee cup. It is the motion of fluids that usually provides mixing. Let us denote \( \theta \) the density of an additive. It satisfies the continuity equation which for an incompressible flow turns into the advection-diffusion equation:

\[
\frac{\partial \theta}{\partial t} = -\text{div} \mathbf{J} = -\text{div} (\mathbf{v} \theta - \kappa \nabla \theta) = - (\mathbf{v} \cdot \nabla) \theta + \text{div} \kappa \nabla \theta .
\] (2.17)

To show that incompressible flows can only increase the flux of \( \theta \), consider two surfaces where we keep different values \( \theta_1, \theta_2 \) and the normal velocity zero (Zeldovich 1937). We assume that all the flux \( J \) generated by one surface is absorbed by another, multiply (2.17) by \( \theta \) and integrate over the space outside the surfaces:

\[
\frac{\partial}{\partial t} \int \theta^2 \, dV = J(\theta_1 - \theta_2) - \int \kappa |\nabla \theta|^2 \, dV .
\] (2.18)

Here the identity \( \theta (\mathbf{v} \cdot \nabla) \theta = (\mathbf{v} \cdot \nabla) \theta^2/2 = \text{div} (\mathbf{v} \theta^2/2) \) made the velocity contribution to disappear. We now consider a steady state or average over time so
that the term with time derivative disappears. Then, for a given $\theta_1, \theta_2$, the flux is proportional to the integral of dissipation:

$$J(\theta_1 - \theta_2) = \int \kappa |\nabla \theta|^2 dV.$$  \hspace{1cm} (2.19)

The minimum of the integral with respect to the variations of $\theta$ that vanish on the boundaries is achieved by the solution of the equation $\text{div} \kappa \nabla \theta = 0$, which corresponds to time-independent or time-averaged solution of (2.17) with zero velocity.\(^8\)

The simplest example of the enhancement of the molecular diffusion due to its interplay with a flow is given by spreading of an additive as it moves along narrow pipes and channels containing uni-directional laminar flow. Consider timescales much exceeding the time of diffusion across the pipe, $a^2/\kappa$, where $a$ is the pipe radius or the channel half-width. The velocity and the concentration are non-uniform across the pipe, which brings extra diffusion along the pipe. The diffusivity can be estimated as the product of the velocity difference between the center and the wall, which we denote $U$, and the size of the region of inhomogeneity of $\theta$. As the spot of $\theta$ spreads along the pipe, molecular diffusion makes it uniform except two intervals at each end, which appeared most recently, during the time not exceeding $a^2/\kappa$. The size of these regions of inhomogeneity can thus be estimated as $Ua^2/\kappa$. Multiplying this length by $U$ we obtain an addition to the diffusivity of order $U^2a^2/\kappa$ (Taylor 1953). That estimate can be validated by deriving from (2.17) the equation that describes the diffusion along the pipe, see Exercise 2.3. The dimensionless parameter $Pe = Ua/\kappa$ is called the Peclet number, it measures relative importance of flow and molecular diffusion.

The effective diffusivity $\kappa + kU^2a^2/\kappa$ as a function of $\kappa$ has a minimum. Here $k$ is a dimensionless factor determined by the geometry of the channel, for a circular tube $k = 1/192$. You can now estimate how fast a drop of medicine spreads in your bloodstream after intravenous injection: taking $a = 0.2\text{cm}$, $U = 0.5 \text{cm/s}$ and $\kappa \simeq 10^{-5} \text{cm}^2/\text{s}$, we obtain $Pe = 10^4$, that is spreading is dominated by Taylor dispersion and the effective diffusivity is $U^2a^2/192\kappa = \kappa Pe^2/192 \simeq 10\text{cm}^2/\text{s}$. In one second the drop shifts by 0.25 cm and spreads approximately by 3 cm.

Another example of diffusivity enhancement at large $Pe$ is seen for diffusion through vortices, for instance, convective cells, which we also characterize by their velocity $U$ and size $a$. To pass to the next cell, the substance must diffuse across a separatrix. The width $\ell$ of the boundary layer across the separatrix can be estimated requiring the diffusion time $\ell^2/\kappa$ to be of order of the turnover time $a/U$, which gives $\ell \simeq (\kappa a/U)^{1/2}$. Since only fluid particles from the boundary layers are able to cross to the next cell and travel far (see the Figure), then
that width plays the role of the mean free path, and the enhanced diffusivity is
\[ U\ell \simeq \sqrt{\kappa Ua} = \kappa Pe^{1/2}. \]
2 Unsteady flows
2.2 Turbulence

Let us consider now turbulent velocity field \( v(\mathbf{r}, t) \) and ask how far a spot of \( \theta \) may deviate from the mean flow during time \( t \). The coordinate of the spot center satisfies the equation

\[
\frac{dq}{dt} = v[q(t), t],
\]

whose solution is the integral of the Lagrangian velocity \( V(t) = v[q(t), t] \) over time: \( q(t) = q(0) + \int_0^t V(t') \, dt' \). If the correlation time \( \tau_c \) of \( V(t) \) is finite, then at \( t \gg \tau_c \) the variance grows by the diffusion law \( \langle q_i(t) g_j(t) \rangle = 2D_{ij} t \) where the so-called eddy diffusivity is as follows

\[
D_{ij} = \frac{1}{2} \int_0^\infty \langle V_i(0)V_j(s) + V_j(0)V_i(s) \rangle \, ds.
\]

Apart from random wandering, the spot also spreads. The spreading of the spot depends on how fast fluid particles separate, which is determined by the dependence of their relative velocity \( \Delta v \) on the separation between them, \( R = q_1 - q_2 \). Two qualitatively different classes of flows must be distinguished: spatially smooth with \( \Delta v(R) \propto R \) and non-smooth with \( \Delta v(R) \propto R^{1-\alpha} \) with \( 0 < \alpha < 1 \). As we learned in Section 2.2.1, turbulent flows are spatially smooth at the scales smaller than the viscous scale \( l(\text{viscous interval}) \) and non-smooth at larger scales (inertial interval), where \( \Delta v(l) \propto l^{1/3} \). On the viscous scale, \( \Delta v(l) \simeq v \), so that the Peclet number is \( v/\kappa \). That ratio is called Schmidt number or Prandtl number when \( \theta \) is temperature. Even though the momentum and substance diffusivities are caused by the same molecular motion, their ratio widely varies depending on the type of material (see also Exercise 2.2). The Schmidt number \( v/\kappa \) is very high for viscous liquids and also for colloids and aerosols since the diffusivity of, say, micron-size cream globules in milk and smoke in the air is six-seven orders of magnitude less than the viscosity of the ambient fluid. In those cases, flows dominate spreading.

At the scales less than \( l \), the velocity difference can be linearized in distance i.e. completely characterized by the matrix of the velocity derivatives \( \nabla_i v_k \), as in Section ?? and Exercise 2.1. When the antisymmetric (vorticity) part of this matrix dominates, fluid elements rotate. In strain-dominated regions fluid elements are deformed: stretched and contracted. The net result of a long sequence of random events of stretching, contraction and rotation is turning any ball into an elongated ellipsoid. The physical reason for it is that substantial deformation appears sooner or later. To reverse it, one needs to contract the long axis of the ellipsoid, that is the direction of contraction must be inside the narrow angle defined by the eccentricity. It is less likely to meet a deformation
inside rather than outside the angle, so that randomly oriented deformations on average continue to increase the eccentricity.

The directions that contract are eventually stopped by molecular diffusion. Since the exponentially growing directions continue to expand then the volume grows exponentially and the value of $\theta$ inside the spot decays exponentially in time. For an arbitrary large-scale initial distribution of $\theta$, the concentration variance decays exponentially in a spatially smooth flow since this is how fast velocity inhomogeneity contracts $\theta$ “feeding” molecular diffusion which eventually decreases the variance. At $Pe \gg 1$, even though it is diffusion that diminishes $\theta$, the rate of decay is usually of order of the typical velocity gradient that is independent of $\kappa$. Recall that vorticity and magnetic field satisfy the same equation (??) as the distance between two close fluid particles, so they can also be exponentially stretched; in particular, this is the mechanism of magnetic dynamo in the Earth core and interstellar gas.

At the scales larger than $l$, in the inertial interval of turbulence cascade, the velocity difference (2.8) scales as $\delta v(r) \propto r^{1/3}$ and the particles separate according to the Richardson’s law $R(t) \propto t^{3/2}$, as we learnt in Section 2.2.1. This is faster than both diffusion ($R \propto t^{1/2}$) and ballistics ($R \propto t$). The reason for it is a multi-scale nature of turbulence, that is the presence of vortices of different sizes. We expect only vortices with $r < R$ to participate in separation, since larger vortices just sweep both particles together. As the particles separate, vortices of larger $r$ and larger $\delta v(r)$ participate and accelerate separation. The volume of any spot also grows so that scalar variance decays by a power law: $\langle \theta^2 \rangle \propto t^{3d/2}$, where $d$ - space dimensionality.

Since the velocity difference (2.8) increases with the distance slower than linearly, then the velocity in turbulence is non-Lipschitz on average, see Section ??, so that fluid trajectories are not well-defined in the inviscid limit.10

Considering two fluid particles whose relative velocity has a negative projection on the line connecting them, and solving the equation $R(t) = R(t_0) - 3Ct/2$, which suggests that the trajectories may intersect in a finite time. Does that mean that if we create a smooth incompressible initial flow with large $Re$, appearance of turbulence will lead to velocity non-smoothness in a finite time? Can we claim our Clay prize now? Not just yet. In such a local flow, the current Reynolds number $R e(\ell) / \nu \propto R^{3/2}$ decreases with the distance so that the viscosity will stop this at the scale $l$ where cascade stops. But Richardson’s law and (2.8) are expected to describe some mean properties of turbulent flows. Maybe finite-time singularities appear in rare fluctuations that do not contribute the cascade? Can we imagine such local flow configuration that compresses energy into a region of diminishing size $\ell(t)$ keeping the Reynolds number $v \ell / \nu$ large? Requiring the energy (rather than the energy flux) constant, we obtain $v^2 \ell^d = E = \text{const}$. That suggests $v = E^{1/2} \ell^{-d/2}$, so that $d \ell / dt = -v$ gives another law of turning the scale into zero: $\ell^{1+d/2}(t) = \ell^{1+d/2}(0) - (1 + d/2)E^{1/2}t$.

Let us check if the viscosity can stop such a contraction: $Re(t) = v(t) \ell(t) / \nu \propto \ell^{1-d/2}$. 
We see that for \( d = 3 \) the Reynolds number is expected to grow so that the viscosity is getting irrelevant and cannot stop it (the case of \( d = 2 \) is marginal in this respect, but it has extra conservation laws which make finite-time singularity impossible). We thus see that the energy conservation does not forbid a finite-time singularity in three dimensions, but so far nobody was able to describe such a flow (or show that it is impossible).

While incompressible flows generally mix, compressible flows can segregate. In other words, \( \theta \) differences are decreased by the former while can be increased by the latter. We shall consider dynamics of compressible flows in the next section, here we make few remarks on kinematics of the fluid particles and substances they carry. The continuity equation written in a Lagrangian form in the reference frame moving with the flow is as follows

\[
\frac{d\theta}{dt} = -\theta \text{div} \mathbf{v} , \quad \ln \left[ \frac{\theta(t)}{\theta(0)} \right] = \int_0^t \text{div} \mathbf{v}(t') \, dt' = C(t) . \tag{2.20}
\]

Volume conservation means that compression factor averaged over the whole flow is zero: \( \langle C \rangle = 0 \). However, the concavity of the exponential function means that the average exponent of it is larger than unity: \( \langle \theta(t)/\theta(0) \rangle = \langle e^C \rangle \geq 1 \). This is because the parts of the flow with positive \( C(t) \) give more contribution into the exponent than the parts with negative \( C(t) \). Moreover, for a fluctuating flow, \( \langle \theta(t)/\theta(0) \rangle \) generally grows with time. Indeed, if the Lagrangian quantity \( \text{div} \mathbf{v}(t) \) is a random function with a zero mean and finite correlation time, then at longer times the logarithm of the density must have a Gaussian statistics with a zero mean and variance linearly growing with time. That means that fluctuations grow, so that the distribution is getting more and more nonuniform tending to a fractal smeared by molecular diffusion\(^{11} \). This is to be contrasted with the mixing and decay of inhomogeneities in an incompressible flow with molecular diffusion.

An interesting and important example of a compressible flow is that of a cloud of particles suspended in a fluid, for instance, rain and cloud droplets in the air. When there are many such particles we may consider the set of their velocities as a field \( \mathbf{u}(\mathbf{r}, t) \) and treat the equation (\( \text{??} \)) as a partial differential equation relating this field to the fluid velocity field \( \mathbf{v}(\mathbf{r}, t) \):

\[
\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = \frac{\mathbf{v} - \mathbf{u}}{\tau} . \tag{2.21}
\]

Even when \( \text{div} \mathbf{v} = 0 \) one obtains for \( \sigma(t) = \text{div} \mathbf{u} \)

\[
\frac{d\sigma}{dt} + \frac{\sigma}{\tau} = -\nabla_i v_k \nabla_k v_i = -\frac{1}{4} \left[ (\nabla_i v_k + \nabla_k v_i)^2 - (\nabla_i v_k - \nabla_k v_i)^2 \right] = \omega^2 - S^2 . \tag{2.22}
\]
We see that the expansion rate $\sigma$ is positive in vorticity-dominated elliptic regions. That means that droplets are thrown out of the air vortices by centrifugal force. The droplets concentrate in strain-dominated hyperbolic regions. Segregation of inertial particles in random flows have many consequences, from planet formation to rain initiation.

Note that the above considerations assumed randomness for Lagrangian quantities, which does not necessarily require flow fluctuating in time, it is enough to have flow pattern complicated enough to provide for the so-called Lagrangian chaos. Indeed, the system of $d$ nonlinear ordinary differential equations, $dq/dt = v(q)$, for generic $v(q)$ has a complicated phase portrait with stochastic attractors.

2.3 Acoustics

Another set of unsteady phenomena is related to the finiteness of the speed of sound $c$. We first consider fluid velocity $v$ much less than $c$ and describe linear acoustics (in the first order in $v/c$). We then account for small non-linearity and dissipation and introduce the Burgers equation. Finally, we consider phenomena that appear when $v$ exceeds $c$.

2.3.1 Sound

Small perturbations of density in an ideal fluid propagate as sound waves that are described by the continuity and Euler equations linearized with respect to the perturbations $p' \ll p_0$, $\rho' \ll \rho_0$:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \text{div} v = 0, \quad \frac{\partial v}{\partial t} + \nabla p'_{\rho_0} = 0. \quad (2.23)$$

To close the system we need to relate the variations of the pressure and density, i.e. specify the equation of state. The derivative of the pressure with respect to the density has the dimensionality of velocity squared so we denote it $c^2$, then $p' = c^2 \rho'$. As noticed in Sect. ??, small oscillations are potential so we introduce $v = \nabla \phi$ and get from (2.23)

$$\phi_{tt} - c^2 \Delta \phi = 0. \quad (2.24)$$

We see that indeed $c$ is the velocity of sound. What is left to establish is what kind of derivative $\partial p / \partial \rho$ one uses, isothermal or adiabatic, which differ substantially.
For a gas, the isothermal derivative gives $c^2 = P/\rho$ while the adiabatic law $P \propto \rho^{\gamma}$ gives:
\[ c^2 = \left( \frac{\partial p}{\partial \rho} \right)_s = \frac{\gamma p}{\rho}. \tag{2.25} \]

One uses an adiabatic equation of state when one can neglect the heat exchange between compressed (warmer) and expanded (colder) regions. This means that the thermal diffusivity (estimated as thermal velocity times the mean free path) must be less than the sound velocity times the wavelength. Since the sound velocity is of the order of the thermal velocity, it requires the wavelength to be longer than the mean free path, which is always so. Therefore, sound must be always treated as adiabatic. Newton already knew that $c^2 = \partial p/\partial \rho$. Experimental data from Boyle showed $p \propto \rho$ (i.e. they were isothermal), which suggested for air $c^2 = p/\rho \simeq 290 \text{ m s}^{-1}$, well off the observed value of $340 \text{ m s}^{-1}$ at $20^\circ \text{C}$. It was only a hundred years later that Laplace got the true (adiabatic) value with $\gamma = 7/5$.

All velocity components, pressure and density perturbations also satisfy the wave equation (2.24). A particular solution of this equation is a monochromatic plane wave, $\phi(\mathbf{r}, t) = \cos(i\mathbf{k r} - i\omega t)$. The relation between the frequency $\omega$ and the wavevector $\mathbf{k}$ is called the dispersion relation; for acoustic waves it is linear: $\omega = ck$. In one dimension, the general solution of the wave equation is particularly simple:
\[ \phi(x, t) = f_1(x - ct) + f_2(x + ct), \tag{2.26} \]
where $f_1$, $f_2$ are given by two initial conditions, for instance, $\phi(x, 0)$ and $\phi_t(x, 0)$. Note that only $v_x = \partial \phi/\partial x$ is non-zero so that sound waves in fluids are longitudinal. Any localized 1D initial perturbation (of density, pressure or velocity along $x$) thus breaks into two plane-wave packets moving in opposite directions without changing their shape. In every such packet, $\partial/\partial t = \pm c \partial/\partial x$ so that the second equation (2.23) gives
\[ v = p'/\rho c = c\rho' / \rho. \tag{2.27} \]

The wave amplitude is small when $\rho' \ll \rho$, which requires $v \ll c$. The (fast) pressure variation in a sound wave, $p' \simeq \rho \omega c$ is much larger than the (slow) variation $\rho \nu^2/2$ one estimates from the Bernoulli theorem.
Considering spherically symmetric case in \( d \) dimensions, one finds that the equation

\[
\phi_{tt} = \frac{c^2}{r^{d-1}} \frac{\partial}{\partial r} \left( r^{d-1} \frac{\partial \phi}{\partial r} \right)
\]  

(2.28)

turns into

\[
h_{tt} = \frac{c^2}{r^2} \frac{\partial^2 h}{\partial r^2}
\]

with the substitution \( \phi = h/r^a \) only for \( d = 1, a = 0 \) and \( d = 3, a = 1 \). We can thus find the general solution of (2.28) in three dimensions as well:

\[
\phi(r, t) = r^{-1} f_1(r - ct) + f_2(r + ct).
\]  

(2.29)

For the case of axial symmetry see the Problem 2.9.

The energy density of sound waves can be obtained by expanding \( \rho E + \rho v^2/2 \) up to the second-order terms in perturbations. The zero-order term \( \rho_0 E_0 \) is constant and thus unrelated to waves. The first-order term \( \rho' (\rho E) / \partial \rho = w_0 \rho' \) is related to the mass redistribution. While this term is generally nonzero, it disappears after integration over the whole volume, so we omit it from the wave energy. We are left with the quadratic terms:

\[
E_w = \frac{\rho_0 v^2}{2} + \frac{\rho'^2}{2} \frac{\partial^2 (\rho E)}{\partial \rho^2} = \frac{\rho_0 v^2}{2} + \frac{\rho'^2}{2} \left( \frac{\partial w_0}{\partial \rho} \right)_s = \frac{\rho_0 v^2}{2} + \frac{\rho'^2 c^2}{2\rho_0}.
\]

The energy flux with the same accuracy is

\[
q = \rho v (w + v^2/2) \approx \rho v w \approx w' \rho_0 v + w_0 \rho' v.
\]

We must omit the energy flux caused by the mass change in a given volume, \( w_0 \rho' v \), since it corresponds to the term \( w_0 \rho' \) omitted in the energy. The enthalpy variation is \( w' = p' (\partial w / \partial p)_s = p' / \rho \approx p' / \rho_0 \) and we obtain

\[
q = p' v.
\]

The energy and the flux are related by \( \partial E_w / \partial t + \text{div} p' v = 0 \). For a plane wave, we obtain from (2.27) \( E_w = \rho_0 v^2 \) and \( q = c E_w \). The energy flux is also called the acoustic intensity. To better hear distant murmur of the brook and be less frightened by sudden lion’s roar, our ear amplifies weak sounds and damp strong ones. It does that by sensing loudness as the logarithm of the intensity for a given frequency. This is why the acoustic intensity is traditionally measured not in watts per square metre but in units of the intensity logarithm, called decibels:

\[
q (\text{dB}) = 120 + 10 \log_{10} q \ (\text{W/m}^2).
\]
Normal conversation in most countries is about 50-60 dB, rock concert is over 100 dB.

The momentum density is
\[ j = \rho v = \rho_0 v + \rho' v = \rho_0 v + q/c^2. \]

Propagating acoustic perturbation of a finite extent in a volume not restricted by walls has a non-zero total momentum \(^{12}\)
\[ \int j \, dV = \rho_0 \int \nabla \phi \, dV + \int q \, dV/c^2 = \int \frac{q \, dV}{c^2}. \] (2.30)

Comment briefly on the momentum of a phonon, which is defined as a sinusoidal perturbation of atom displacements; a monochromatic wave in these (Lagrangian) coordinates has zero momentum. \(^{13}\) But our equations (2.24,2.28) and solutions (2.26,2.29) are written in Eulerian coordinates. A perturbation, which is sinusoidal in Eulerian coordinates, has a non-zero momentum at second order (where Eulerian and Lagrangian coordinates differ). Indeed, let us consider the Eulerian velocity field as a monochromatic wave with a given frequency and a wavenumber: \[ v(x,t) = u \sin(kx - \omega t). \] The Lagrangian coordinate \( X(t) \) of a fluid particle satisfies the following equation:
\[ \dot{X} = v(X,t) = u \sin(kX - \omega t). \] (2.31)

This is a non-linear equation, which can be solved by iteration, \( X(t) = X_0 + X_1(t) + X_2(t) + \ldots \) assuming \( v \ll \omega/k \). We shall see below in Sect. ?? that \( \omega/k \) is the wave phase velocity. When it is much larger than the fluid velocity, the fluid particle displacement during the wave period is much smaller than the wavelength. Such an iterative solution gives oscillations at first order and a mean drift at second order:
\[ X_1(t) = \frac{u}{\omega} \cos(kX_0 - \omega t), \]
\[ X_2(t) = \frac{ku^2 t}{2\omega} - \frac{ku^2}{4\omega^2} \sin(2(kX_0 - \omega t)). \] (2.32)

We see that at first order in wave amplitude the perturbation propagates, while at second order the fluid itself flows in the direction of wave propagation with the speed \( ku^2/2\omega = u^2/2c \). Fluid particles move with the wave a bit longer than against it.

However, nonzero momentum does not necessarily mean net flow in one dimension. Fluid particles exchange momentum, which thus can be transported
Unsteady flows without mass transfer. For example, one can generate the wave train by moving piston in a tube back and forth, without producing any net flow. In this case, every fluid particle oscillates in the wave and returns to its original position after the wave train has passed by it. Therefore, the time average of the mass flux must be zero at any point: \( \overline{\rho v} = \rho_0 \overline{\rho v} + \rho' \overline{\rho v} = 0 \). That requires the mean counterflow, \( \overline{\rho v} = -\overline{\rho v}/\rho_0 = -\overline{v^2}/c = -u^2/2c \), which exactly cancels the Lagrangian drift (2.32), derived under the assumption of no mean Eulerian velocity. The lesson is that the drift is a quadratic quantity and must be determined by using the Eulerian velocity valid up to quadratic terms as well. While the momentum \( \overline{\rho v} \) averaged over time at a given point is zero, the momentum (2.30) averaged over space at a given time is nonzero.

### 2.3.2 Riemann wave

As we have seen, an infinitesimally small one-dimensional acoustic perturbation splits into two simple waves, which then propagate without changing their forms. Let us show that such purely adiabatic waves of a permanent shape are impossible for finite amplitudes (Earnshaw paradox): in the reference frame moving with speed \( c \) one would have a steady motion with the continuity equation \( \rho v = \text{const.} = C \) and the Euler equation \( \rho dv = -d p/\rho \), giving \( dp/d\rho = (C/\rho)^2 \) and \( p = p_0 - C^2/\rho \), which contradicts the second law of thermodynamics (such gas would have negative pressure for sufficiently large momentum \( C \) or sufficiently small density). It is thus clear that a simple plane wave must change under the action of a small factor of non-linearity.

Consider now a one-dimensional motion of a compressible fluid having the usual adiabatic equation of state \( p = p_0 (\rho/\rho_0)^\gamma \) with \( \gamma > 1 \). Let us look for a simple wave where one can express any two of \( v, p, \rho \) via the remaining one. This is a generalization for a non-linear case of what we did for a linear wave (only now it must be non-stationary). Suppose that we assume everything to be determined by \( v \), that is \( p(v) \) and \( \rho(v) \). The Euler and continuity equations take the form:

\[
\frac{dv}{dt} = -\frac{1}{\rho} c^2(v) \frac{d\rho}{dv} \frac{\partial v}{\partial x}, \quad \frac{d\rho}{dt} \frac{dv}{dv} = -\rho \frac{\partial v}{\partial x}. \tag{2.33}
\]

Here \( c^2(v) \equiv dp/d\rho \). Excluding \( d\rho/dv = \pm \rho/c \), one gets

\[
\frac{dv}{dt} = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \pm c(v) \frac{\partial v}{\partial x}. \tag{2.34}
\]
Two signs correspond to waves propagating in opposite directions. In a linear approximation we had \( u_t + cu_x = 0 \), where \( c = \sqrt{\gamma p_0 / \rho_0} \). Now we find

\[
c(v) = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\rho_0 + \delta p}{\rho_0 + \delta \rho}} = c \left(1 + \frac{\delta p}{2\rho_0} - \frac{\delta \rho}{2\rho_0}\right) = c + v\gamma - \frac{1}{2},
\]

since \( \delta \rho / \rho_0 = v / c \). The local sound velocity increases with amplitude since \( \gamma > 1 \), that is the positive effect of the pressure increase overcomes the negative effect of the density increase.

Taking a plus sign in (2.34) we get the equation for the simple wave propagating rightwards

\[
\frac{\partial v}{\partial t} + \left(c + v\gamma + \frac{1}{2}\right)\frac{\partial v}{\partial x} = 0.
\]

This equation describes the simple fact that the higher the amplitude of the perturbation the faster it propagates, both because of higher velocity and of higher pressure gradient. (J.S. Russel remarked in 1885 that ‘the sound of a cannon travels faster than the command to fire it’.) Since different parts of the wave profile move with different speeds, then the profile changes. In particular, faster fluid particle will catch up with slower moving particles. Indeed, if we have the initial distribution \( v(x, 0) = f(x) \), the solution of (2.36) is given by an implicit relation

\[
v(x, t) = f\left[x - \left(c + v\gamma + \frac{1}{2}\right)t\right],
\]

which can be useful for particular \( f \) but is not much help in a general case. An explicit solution can be written in terms of characteristics (the lines in the \( x-t \) plane that correspond to constant \( v \), Figure 2.10):

\[
\frac{\partial x}{\partial t} = c + v\gamma + \frac{1}{2} \quad \Rightarrow x = x_0 + ct + \frac{\gamma + 1}{2}v(x_0)t, \tag{2.38}
\]

where \( x_0 = f^{-1}(v) \). The solution (2.38) is called a simple or Riemann wave.

In the variables \( \xi = x - ct \) and \( u = v(\gamma + 1)/2 \), the equation takes the form

\[
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial \xi} = \frac{du}{dt} = 0,
\]
which describes freely (inertially) moving particles. Indeed, we see that the characteristics are straight lines with the slopes given by the initial distribution $v(x, 0)$, that is every fluid particle propagates with a constant velocity. It is seen that the parts where $\partial v(x, 0)/\partial x$ was initially positive will decrease their slope while the negative slopes in $\partial v(x, 0)/\partial x$ become steeper (Figure 2.11).

The characteristics are actually Lagrangian coordinates: $x(x_0, t)$. The characteristics cross in the $x-t$ plane (and particles hit each other) when $(\partial x/\partial x_0)_t$ turns into zero, that is

$$1 + \gamma + \frac{1}{2} \frac{dv}{dx_0} t = 0,$$

which first happens with particles that correspond to $dv/dx_0 = f'(x_0)$ maximal negative, that is $f''(x_0) = 0$. When characteristics cross, we have different velocities at the same point in space, which corresponds to a discontinuity in the velocity field called shock.

A general remark: notice the qualitative difference between the properties of the solutions of the hyperbolic equation $u_{tt} - c^2 u_{xx} = 0$ and the elliptic equations, say the Laplace equation. As mentioned in Section ??, elliptic equations have solutions and their derivatives which are regular everywhere inside the domain of existence. On the contrary, hyperbolic equations propagate perturbations along the characteristics and characteristics can cross (when $c$ depends on $u$ or $x$, $t$), leading to singularities.

### 2.3.3 Burgers equation

We thus see that in ideal hydrodynamics non-linearity makes the propagation velocity depend on the amplitude, which leads to crossing of characteristics and

![Figure 2.10](image-url)
thus to wave breaking: any acoustic perturbation tends to create a singularity (shock) in a finite time. An account of spatial derivatives higher than the first ones (that enter the equations of ideal hydrodynamics) is necessary near a shock. In this section, we account for the next derivative (the second one), which corresponds to viscosity:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}.
\]

(2.39)

This is the Burgers equation, the first representative of the small family of universal non-linear equations (we shall meet two other equally famous members, the Kortevge-de-Vries and non-linear Schrödinger equations, in the next chapter, where we account, in particular, for the third derivative in acoustic-like perturbations). Note that \( \nu \) in the Burgers equation is half the kinematic viscosity. Indeed, from the linearized Navier-Stokes and continuity equations one can readily obtain for the frequency of sound:

\[
\omega = \sqrt{c^2 k^2 - i \nu k^3 c} \approx c k - i \nu k^2 / 2.
\]

The Burgers equation is a minimal model of fluid mechanics: a single scalar field \( u(x, t) \) changes in one dimension under the action of inertia and friction. This equation describes wide classes of systems with hydrodynamic-type non-linearity, \( (u \nabla) u \), and viscous dissipation. It can be written in a potential form \( u = \nabla \phi \); then \( \phi_t = - (\nabla \phi)^2 / 2 + \nu \Delta \phi \); in such a form it can be considered in one and two spatial dimensions, where it describes in particular the surface growth under uniform deposition and diffusion; the deposition contribution into the time derivative of the surface height \( \phi(r) \) is proportional to the flux per unit area, which is inversely proportional to the area, \( 1 + (\nabla \phi)^2 - 1 / 2 \approx 1 - (\nabla \phi)^2 / 2 \), as shown in Figure 2.12.

The Burgers equation can be linearized by the Hopf substitution \( u = -2 \nu \phi / \psi \):

\[
\frac{\partial}{\partial \xi} \left( \frac{\phi_t - \nu \phi_{\xi \xi} \psi}{\psi} \right) = 0 \Rightarrow \phi_t - \nu \phi_{\xi \xi} = \phi C'(t),
\]
which with the change \( \phi \rightarrow \phi \exp C \) (not changing \( u \)) gives the linear diffusion equation:

\[
\phi_t - \nu \phi_{\xi \xi} = 0.
\]

The initial value problem for the diffusion equation is solved as follows:

\[
\phi(\xi, t) = \frac{1}{\sqrt{4\pi \nu t}} \int_{-\infty}^{\infty} \phi(\xi', 0) \exp \left[ -\frac{(\xi - \xi')^2}{4\nu t} \right] d\xi' \quad (2.40)
\]

Despite the fact that the Burgers equation describes a dissipative system, it conserves total momentum (as any viscous equation does).\(^{16}\) If the momentum is finite, then any perturbation evolves into a universal form depending only on \( M = \int u(x) \, dx \) and not on the form of \( u(\xi, 0) \). At \( t \rightarrow \infty \), (2.40) gives \( \phi(\xi, t) \rightarrow \pi^{-1/2} F(4\nu t)^{-1/2} \), where

\[
F(y) = \int_{-\infty}^{\infty} \exp \left[ -\frac{y^2}{2\nu} - \frac{1}{2\nu} \int_{0}^{(y-\eta)\sqrt{4\nu t}} u(\eta', 0) \, d\eta' \right] \, d\eta
\approx e^{-M/4\nu} \int_{-\infty}^{y} e^{-\eta^2} \, d\eta + e^{M/4\nu} \int_{y}^{\infty} e^{-\eta^2} \, d\eta. \quad (2.41)
\]

Solutions with positive and negative \( M \) are related by the transform \( u \rightarrow -u \) and \( \xi \rightarrow -\xi \).

---

**Figure 2.12** If the \( x \)-axis is along the direction of the local surface change then the local area element is \( df = \sqrt{(dx)^2 + (d\phi)^2} = dx \sqrt{1 + (\nabla \phi)^2} \).
2.3 Acoustics

\[ u \sim (2Mt)^{1/2} \xi \]

Note that \( \frac{M}{\nu} \) is the Reynolds number and it does not change while the perturbation spreads. This is a consequence of momentum conservation in one dimension. In a free viscous decay of a \( d \)-dimensional flow, usually the scale grows as \( R(t) \propto t^{1/2} \). To keep the momentum, velocity must decay as \( R^{-d} \propto t^{-d/2} \) so that the Reynolds number evolves as \( t^{(1-d)/2} \). For a wake behind the body (see Sect. 2.2.2), the Reynolds number does not change at \( d = 2 \). For a jet in a fluid (see Sect. ??), constancy of the momentum flux \( \int u^2 \, dt \) means that the Reynolds number \( uR/\nu \) does not change along the jet at \( d = 3 \).

When \( M/\nu \gg 1 \) the solution looks particularly simple, as it acquires a saw-tooth form. Indeed, in the interval \( 0 < y < M/2 \) (i.e. for \( 0 < \xi < \sqrt{2Mt} \)) the first integral in (2.41) is negligible and \( F \sim \exp(-y^2) \) so that \( u(\xi, t) = \xi/t \). For both \( \xi < 0 \) and \( \xi > \sqrt{2Mt} \) we have \( F \sim \text{const} + \exp(-y^2) \) so that \( u \) is exponentially small there.

An example of the solution with an infinite momentum is a steady propagating shock. Let us look for a travelling wave solution \( u(\xi - wt) \). In this case the Burgers equation is reduced to an ordinary differential equation which can be immediately integrated to give \( -uw + u^2/2 = vu_\xi \) under the assumption that \( u \to 0 \) at least for one of the infinities. Integrating again:

\[ u(\xi, t) = \frac{2w}{1 + C \exp(w(\xi - wt)/\nu)} \quad (2.42) \]

We see that this is a shock having width \( v/w \) and propagating with velocity that is half the velocity difference on its sides. A simple explanation is that the shock front is the place where a moving fluid particle hits a standing fluid particle, they stick together and continue with half velocity owing to momentum conservation. The form of the shock front is steady since non-linearity is balanced by viscosity.
The Burgers equation is Galilean invariant, that is if \( u(\xi, t) \) denotes a solution so does \( u(\xi - w t) + w \) for an arbitrary \( w \). In particular, one can transform (2.42) into a standing shock, \( u(\xi, t) = -w \tanh(w \xi / 2 \nu) \).

### 2.3.4 Acoustic turbulence

The shock wave (2.42) dissipates energy at the rate \( \nu \int u^2 \, dx \) independent of viscosity, see (2.43). Indeed, when a moving particle sticks to a standing particle, half of their kinetic energy is lost, independently of how sticking actually occurs. In compressible flows, shock creation is a way of dissipating finite energy in the inviscid limit (in incompressible flows, this was achieved by turbulent cascade). The solution (2.42) shows how it works: the velocity derivative goes to infinity as the viscosity goes to zero. In the inviscid limit, the shock is a velocity discontinuity.

Consider now acoustic turbulence produced by a pumping correlated on much larger scales, for example, pumping a pipe from one end by frequencies \( \Omega \) much less than \( cv / \nu \), so that the Reynolds number is large. Upon propagation along the pipe, such turbulence evolves into a set of shocks at random positions with the mean distance between shocks \( L \approx cv / \nu \) far exceeding the shock width \( \nu / w \), which is a dissipative scale. For every shock (2.42),

\[
S_3(x) = \frac{1}{L} \int_{-L/2}^{L/2} u(x + x') - u(x') \, dx' \approx -8w^3 x / L ,
\]

\[
\epsilon = \frac{1}{L} \int_{-L/2}^{L/2} \nu u_x^2 \, dx \approx 2w^3 / 3L , \tag{2.43}
\]

which gives:

\[
S_3 = -12\epsilon x . \tag{2.44}
\]

This formula is a direct analogue of the flux law (2.11). As in Section 2.2.1, it would be wrong to assume \( S_n = \langle u(x) - u(0)^n \rangle \simeq (\epsilon x)^{n/3} \), since shocks give a
2.3 Acoustics

much larger contribution for \( n > 1 \): \( S_n \approx w^n x / L \), here \( x / L \) is the probability of finding a shock in the interval \( x \).

In terms of Fourier harmonics, every shock contributes \( u_k \propto 1/k \), which indeed gives \( S_2(x) = \langle u(x) - u(0)^2 \rangle = \int |u_k|^2 (1 - e^{ikx}) \, dk \propto \int 1/|u_k|^2 \, dk \propto x \).

Generally, \( S_n(x) \propto C_n|x|^n + C'_n|x| \), where the first term comes from the smooth parts of the velocity (the right \( x \)-interval in Figure 2.13) while the second comes from \( O(x) \) probability having a shock in the interval \( x \).

The scaling exponents, \( \xi_n = d \ln S_n / d \ln x \), thus behave as follows: \( \xi_n = n \) for \( 0 \leq n \leq 1 \) and \( \xi_n = 1 \) for \( n > 1 \). Like incompressible (vortex) turbulence in Section 2.2.1, this means that the probability distribution of the velocity difference \( P(\Delta u, x) \) is not scale-invariant in the inertial interval, that is the function of the re-scaled velocity difference \( \Delta u / x^\epsilon \) cannot be made scale-independent for any \( \epsilon \). The simple bi-modal nature of Burgers turbulence (shocks and smooth parts) means that the PDF is actually determined by two (non-universal) functions, each depending on a single argument: \( P(\Delta u, x) = f_1(\Delta u / x) + f_2(\Delta u / u_{rms}) \). The breakdown of scale invariance means that the low-order moments decrease faster than the high-order ones as one goes to smaller scales. That means that the level of fluctuations increases with the resolution: the smaller the scale the more probable are large fluctuations. When the scaling exponents \( \xi_n \) do not lie on a straight line, this is called an anomalous scaling since it is related again to the symmetry (scale invariance) of the PDF broken by pumping and not restored even when \( x / L \to 0 \).

Alternatively, one can derive the equation on the structure functions similar to (2.10):

\[
\frac{\partial S_2}{\partial t} = -\frac{3}{3x} \frac{\partial S_3}{\partial x} - 4\epsilon + 6 \frac{\partial^2 S_2}{\partial x^2} . \tag{2.45}
\]

Here \( \epsilon = \nu \langle u^2 \rangle \). Equation (2.45) describes both a free decay (then \( \epsilon \) depends on \( t \)) and the case of a permanently acting pumping that generates turbulence that is statistically steady for scales less than the pumping length. In the first case, \( \partial S_2 / \partial t \simeq S_2 u / L \ll \epsilon \simeq u^3 / L \) (where \( L \) is a typical distance between shocks) while in the second case \( \partial S_2 / \partial t = 0 \).

In both cases, \( S_3 = -12 \epsilon x + 3v \partial S_2 / \partial x \). Consider now the limit \( \nu \to 0 \) at fixed \( x \) (and

![Figure 2.13 Typical velocity profile in Burgers turbulence.](image)
2 Unsteady flows

$t$ for decaying turbulence). Shock dissipation provides for a finite limit of $\epsilon$ at $\nu \to 0$, which gives (2.44). Similarly to incompressible turbulence, a flux constancy fixes $S_1(x)$, which is universal, determined solely by $\epsilon$ and depends neither on the initial statistics for decay nor on the pumping for steady turbulence. Higher moments can be related to the additional integrals of motion, $E_n = \int u^n \, dx / 2$, which are all formally conserved in the inviscid case. In reality, any shock dissipates the finite amount $\epsilon_n$ of $E_n$ in the limit $\nu \to 0$ so that one can express $S_{2n+1}$ via these dissipation rates for integer $n$: $S_{2n+1} \propto \epsilon_n x$ (see Exercise 2.5). That means that the statistics of velocity differences in the inertial interval depend on the infinitely many pumping-related parameters, the fluxes of all dynamical integrals of motion.

For incompressible (vortex) turbulence described in Section 2.2.1, we have neither understanding of structures nor classification of the conservation laws responsible for an anomalous scaling.

2.3.5 Mach number

Compressibility leads to finiteness of the propagation speed of perturbations. Here we consider the motions (of the fluid or bodies) with velocity exceeding the sound velocity. The propagation of perturbations in more than one dimension is peculiar for supersonic velocities. Indeed, consider fluid moving uniformly with velocity $v$. If a small disturbance appears at some place $O$, it will propagate with respect to the fluid with the sound velocity $c$. All possible velocities of propagation in the rest frame are given by $v + cn$ for all possible directions of the unit vector $n$. This means that in a subsonic case ($v < c$) the perturbation propagates in all directions around the source $O$ and eventually spreads to the whole fluid. This is seen from Figure 2.14, where the left circle contains $O$. However, in a supersonic case, vectors $v + cn$ all lie within a $\alpha$-cone with $\alpha = \arcsin c / v$ called the Mach angle. Outside the Mach cone, shown in Figure 2.14 by broken lines, the fluid stays undisturbed. The dimensionless ratio $v/c = M$
2.3 Acoustics

is called the Mach number, which is a control parameter like Reynolds number, flows are similar for the same $Re$ and $M$.

If sound is generated, say, by periodic pulsations of a source moving relative to the fluid, the circles in Figure 2.15 correspond to the lines of a constant phase. The wavelength (the distance between the constant-phase surfaces) is smaller to the left of the source by the factor $1 - v/c$ because the propagation speed $c - v$ is smaller. For the case of a moving source, this means that the wavelength is shorter in front of the source and longer behind it. For the case of a moving fluid, it means that the wavelength is shorter upwind. The frequencies registered by the observer are, however, different in these two cases:

(i) When the emitter and receiver are at rest, the frequencies emitted and received are the same; the wavelength received upwind is smaller by a factor $1 - v/c$ and downwind larger by a factor $1 + v/c$ in a moving fluid.

(ii) When the source moves towards the receiver while the fluid is still, the propagation speed is $c$ and the smaller wavelength corresponds to the frequency received being larger by the factor $1/(1 - v/c)$. This frequency change due to a relative motion of source and receiver is called the Doppler effect.

Doppler effect is used to determine experimentally the fluid velocity by scattering sound or light on particles carried by the flow and by the police to catch us speeding.\(^4\) Doppler radar emits wave with the frequency $\omega_0$ towards a mirror (car) approaching with the speed $v$. In the mirror reference frame, the wave is received and reflected with the frequency $\omega_0(1 + v/c)$; then police detector receives from a moving source the frequency, which is higher by yet another factor $1/(1 - v/c)$, so that the frequency received is $\omega_0(c + v)/(c - v)$.

Imagine a source producing wave maxima every second so that the distance

Figure 2.15 Circles are constant-phase surfaces of an acoustic perturbation generated in a fluid that moves to the right with a subsonic (left) and supersonic (right) speed. Alternatively, this may be seen as sound generated by a source moving to the left.
between them is \( c \). The maxima hit the moving mirror every \( c/(c+v) \) seconds and are reflected back, but every subsequent maximum meets the mirror closer to the source, so the distance between them (wavelength) after reflection is \( c(c-v)/(c+v) \), and they come back to the source with the speed \( c \) every \( (c-v)/(c+v) \) seconds.

Let us now describe the frequency dependence on the direction of propagation. When the fluid moves relative to the receiver, it registers frequency that is different from the frequency \( c k \) measured in the fluid frame. Let us find the relation between the frequencies. The monochromatic wave is \( \exp(ik \cdot r' - ckt) \) in a reference frame moving with the fluid. The coordinates in the moving and rest frames are related as \( r' = r - vt \) so in the rest frame we have \( \exp(ik \cdot r - ckt - k \cdot v t) = \exp(ik \cdot r - \omega_0 t) \), which means that the frequency measured in the rest frame is:

\[
\omega_k = ck + (k \cdot v).
\]

(2.46)

This change of frequency, \( (k \cdot v) \), is called the Döppler shift. When sound propagates upwind, one has \( (k \cdot v) < 0 \), so that a standing person hears a lower tone than those downwind. Another way to put it is that the wave period is larger since more time is needed for a wavelength to pass our ear as the wind sweeps it.

Consider now a wave source that oscillates with frequency \( \omega_0 \) and moves with the velocity \( u \). The frequency registered in the rest frame depends on the direction of propagation. Indeed, to relate \( \omega_0 \) to the frequency in still air \( \omega = ck \), pass to the reference frame moving with the source where \( \omega_k = \omega_0 \) and the fluid moves with \( -u \) so that (2.46) gives

\[
\omega_0 = ck - (k \cdot u) = \omega_1 - (u/c) \cos \theta,
\]

(2.47)

where \( \theta \) is the angle between \( u \) and \( k \). In the fluid reference frame one receives \( \omega = \omega_0/1 - (u/c) \cos \theta \).

Let us now look at (2.46) for \( v > c \). We see that the frequency of sound registered in the rest frame turns into zero on the Mach cone (also called the characteristic surface). The condition \( \omega_k = 0 \) defines in \( k \)-space the cone surface \( ck = -k \cdot v \) or in any plane the relation between the components:

\[
v^2 k_x^2 + v^2 k_y^2 = c^2 k_x^2 + k_y^2.
\]

The propagation fronts of perturbation in the \( x, y \)-plane are determined by the constant-phase condition \( k_x dx + k_y dy = 0 \) and \( dy/dx = -k_x/k_y = \pm c/\sqrt{v^2 - c^2} \), which again corresponds to the broken straight lines in Figure 2.15 with the same Mach angle \( \alpha = \arcsin(c/v) = \arctan(c/\sqrt{v^2 - c^2}) \).

We thus see that there is a stationary perturbation along the Mach surface, with acoustic waves inside it and undisturbed fluid outside. The Mach surface is an
2.3 Acoustics

Figure 2.16 Subsonic (left) and supersonic (right) flow around a slender wing.

example of caustic which is a general term for boundary between a region with no waves and a region with two group of waves. Indeed, two constant-phase surfaces intersect in every point inside the cone. We give general description of caustics in Section sec:group below.

Let us now consider a flow past a body in a compressible fluid. For a slender body, like a wing, the flow perturbation can be considered small, as in Section ??, only now including density: $u + v, \rho_0 + \rho', P_0 + P'$. For small perturbations, $P' = c^2 \rho'$. Linearization of the steady Euler and continuity equations gives

$$
\rho_0 \frac{\partial v}{\partial x} = -\nabla P' = -c^2 \nabla \rho', \quad u \frac{\partial \rho'}{\partial x} = -\rho_0 \text{div} v. \quad (2.48)
$$

Taking the curl of the Euler equation, we get $\partial \omega / \partial x = 0$. Since vorticity is $x$-independent and zero far upstream, it is zero everywhere (in a linear approximation). We thus have a potential flow, $v = \nabla \phi$, which satisfies

$$
(1 - M^2) \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \quad (2.49)
$$

Here the Mach number, $M = u / c$, determines whether the equation is elliptic or hyperbolic. When $M < 1$, the equation is elliptic and the streamlines are everywhere smooth. When $M > 1$, the equation is hyperbolic and the streamlines have cusps on the Mach planes, extending from the ends of the wing; the streamlines are straight outside and curved only between the planes, see Figure 2.16.

In the elliptic case, the change of variables $x \rightarrow x(1 - M^2)^{-1/2}$ turns (2.49) into a Laplace equation, $\text{div} v = \Delta \phi = 0$, which we had for an incompressible case. To put it simply, at subsonic speeds, compressibility of the fluid is equivalent to a longer body. Since the lift is proportional to the velocity circulation, i.e. to the wing length, then we conclude that compressibility increases the lift by $(1 - M^2)^{-1/2}$.

In the hyperbolic case, the solution is

$$
\phi = F(x - By), \quad B = (M^2 - 1)^{-1/2}.
$$
The boundary condition on the body having the shape \( y = f(x) \) is \( v_y = \partial \phi / \partial y = u f'(x) \), which gives \( F = -u f' / B \). This means that \( v_y / u = f'(x) \) everywhere, that is the streamlines repeat the body shape and turn straight behind the rear Mach surface (in the linear approximation). We see that passing through the Mach surface the velocity and density have a jump proportional to \( f'(0) \). This means that Mach surfaces (like the planes or cones described here) are actually shocks. One can relate the flow properties before and after the shock by the conservation laws of mass, energy and momentum, in this case called the Rankine–Hugoniot relations. Namely, if \( w \) is the velocity component normal to the front, then the fluxes \( \rho w, \rho w (W + w^2/2) = \rho w y P / (\gamma - 1) \rho + w^2/2 \) and \( P + \rho w^2 \) must be continuous through the shock. This gives three relations that can be solved for the pressure, velocity and density after the shock (Exercise 2.3). In particular, for a slender body when the streamlines deflect by a small angle \( \delta = f'(0) \) after passing through the shock, we get the velocity changes \( v_y = \partial \phi / \partial y = u \delta \) and \( v_x = \partial \phi / \partial x = -u \delta / B \). The pressure change due to the velocity decrease is as follows:

\[
\frac{\Delta P}{P} \propto \frac{u^2 - (u + v_x)^2 - v_y^2}{c^2} = M^2 \left[ 1 - \left( 1 - \frac{\delta}{\sqrt{M^2 - 1}} \right)^2 - \delta^2 \right] \\
\approx \frac{2 \delta M^2}{\sqrt{M^2 - 1}}. \tag{2.50}
\]

The compressibility contribution to the drag is proportional to the pressure drop and thus the drag jumps when \( M \) crosses unity, owing to the appearance of shock and the loss of acoustic energy radiated away between the Mach planes. The drag and lift singularity at \( M \to 1 \) is sometimes referred to as the ‘sound barrier’. Apparently, our assumption of small perturbations does not work at \( M \to 1 \). For comparison, recall that the wake contribution to the drag is proportional to \( \rho u^2 \), while the shock contribution (2.50) is proportional to \( P M^2 / \sqrt{M^2 - 1} \simeq \rho u^2 / \sqrt{M^2 - 1} \).

We see that in a linear approximation, the steady-state two-dimensional flow perturbation does not decay with distance. We have learnt in Section 2.3.2 that the propagation speed depends on the amplitude and so must the angle \( \alpha \), which means that the Mach surfaces are straight only where the amplitude is small, which is usually far away from the body. Weak shocks with \( M - 1 \ll 1 \) can be described by the Burgers equation. Indeed, according to (2.41) and (2.42), the front width is

\[
\frac{v}{u-c} = \frac{v}{c(M-1)} \simeq \frac{lv}{c(M-1)},
\]
which exceeds the mean free path \( l \) only for \( \mathcal{M} - 1 \ll 1 \) since the molecular thermal velocity \( v_T \) and the sound velocity \( c \) are comparable (see also Section ??). To be consistent in the framework of continuous media, strong shocks must be considered as discontinuities.

Exercises

2.1 (i) Two-dimensional incompressible flow around a saddle-point corresponds to a pure strain: \( v_x = \lambda x, \ v_y = -\lambda y \). The coordinates \( x(t), y(t) \) of a fluid particle satisfy the equations \( \dot{x} = v_x \) and \( \dot{y} = v_y \). Whether the vector \( r = (x, y) \) is stretched or contracted after some time \( T \) depends on its orientation and on \( T \). Find which fraction of the vectors is stretched.

(ii) Consider a two-dimensional incompressible flow having both permanent strain \( \lambda \) and vorticity \( \omega \): \( v_x = \lambda x + \omega y/2, \ v_y = -\lambda y - \omega x/2 \). Describe the motion of the particle, \( x(t), y(t) \), for different relations between \( \lambda \) and \( \omega \).

2.2 Consider a fluid layer between two horizontal parallel plates kept at the distance \( h \) at temperatures that differ by \( \Delta T \). The fluid has kinematic viscosity \( \nu \), thermal conductivity \( \chi \) (both measured in \( \text{cm}^2 \text{s}^{-1} \)) and the coefficient of thermal expansion \( \beta = -\partial \ln \rho / \partial T \), such that the relative density change due to the temperature difference, \( \beta \Delta T \), far exceeds the change due to the hydrostatic pressure difference, \( gh/\bar{c}^2 \), where \( \bar{c} \) is the velocity of sound. Find the control parameter(s) for the appearance of the convective (Rayleigh–Bénard) instability.

2.3 Taylor dispersion along a narrow pipe. Derive the laminar Poiseuille profile carrying mean flux \( U \) in a circular pipe of radius \( a \). Starting from the advection-diffusion equation (2.17), derive the equation on concentration averaged over the cross-section, \( \bar{\theta}(x, t) = \int \theta(x, r, t) \, dr / \pi a^2 \), considering times exceeding \( a^2 / \kappa \), were \( \kappa \) is the molecular diffusivity. Find the effective diffusivity along the pipe.

2.4 Consider a shock wave with the velocity \( w_1 \) normal to the front in a polytropic gas having the enthalpy

\[
W = c_p T = PV \gamma / (\gamma - 1) = \rho \gamma / (\gamma - 1) = c_s^2 / (\gamma - 1),
\]

where \( \gamma = c_p / c_v \). Write Rankine–Hugoniot relations for this case. Express the ratio of densities \( \rho_2 / \rho_1 \) via the pressure ratio \( P_2 / P_1 \), where the subscripts 1 and 2 denote the values before and after the shock. Express \( P_2 / P_1 \),
\( \rho_2 / \rho_1 \) and \( \mathcal{M}_2 = w_2 / c_2 \) via the pre-shock Mach number \( \mathcal{M}_1 = w_1 / c_1 \).

Consider the limits of strong and weak shocks.

2.5 For Burgers turbulence, express the fifth structure function \( S_5 \) via the dissipation rate \( \epsilon = 6 \nu (u^2 u_x^2) + (u_x^2) (u_x^2) \).

2.6 In a standing sound wave, air with density \( \rho \) locally moves as follows: \( v = a \sin(\omega t) \). Consider a small suspended spherical particle with density \( \rho_0 \) whose material evaporates so that its volume \( V(t) \) decreases with the rate proportional to the relative velocity: \( V^{-1} dV/dt = -\alpha |v - u| \). Find how the particle velocity \( u \) changes with time if it was initially at rest. Assume \( \rho_0 \gg \rho \).

2.7 There is anecdotal evidence that early missiles suffered from an interesting malfunction of the fuel gauge. The gauge was a simple floater (a small air-filled rubber balloon) whose position was supposed to signal the level of liquid fuel during the ascending stage. However, when the engine was warming up before starting, the gauge unexpectedly sank to the bottom, signalling zero level of fuel and shutting off the engine. How do engine-reduced vibrations reverse the sign of effective gravity for the balloon in the fluid?

---

\[ \text{Consider an air bubble in the vessel filled up to the depth } h \text{ by a liquid with density } \rho. \text{ The vessel vibrates vertically according to } x(t) = \left( \frac{Ag}{\omega^2} \right) \sin(\omega t), \text{ where } g \text{ is the static gravity acceleration. Find the threshold amplitude } A \text{ necessary to keep the bubble near the bottom. The pressure on the free surface is } P_0. \]

2.8 It is a common experience that acoustic intensity drops quickly with the distance when the sound propagates upwind. Why is it so difficult to hear somebody shouting against the wind?

2.9 Describe an axial symmetric propagation of acoustic waves. Hint: use (2.29).
2.3 Acoustics

2.10 The relation (2.47), \( \omega_0 = \omega_1 - (u/c) \cos \theta \), suggests that the frequencies \( \omega_0 \) (emitted) and \( \omega \) (received) have different signs when a sound source moves towards a receiver with a supersonic speed, \( u \cos \theta > c \). What does it mean physically?

2.11 Consider a steady flow of a compressible ideal fluid with no external body force. Does the flux \( \rho v \) along the streamline increase or decrease with the velocity \( v \)? Hint: express pressure variation as \( dp = c^2 dp \).

2.12 Consider a spherically symmetric radial steady flow of an isothermal ideal gas in the gravity field of a star with mass \( M \) (a model of stellar wind). Can the velocity of such a flow grow with the distance?