3

Dispersive waves

In this chapter, we consider systems that support small-amplitude waves whose speed depends on wavelength. This is in distinction from acoustic waves (or light in the vacuum) that all move with the same speed so that a small-amplitude one-dimensional perturbation propagates without changing its shape. When the speeds of different Fourier harmonics are different, the shape of a perturbation generally changes as it propagates. In particular, initially localized perturbation spreads. That is, dispersion of wave speed leads to packet dispersion in space. This is why such waves are called dispersive. Since different harmonics move with different speeds, then they separate with time and can subsequently be found in different places. As a result, for quite arbitrary excitation mechanisms one often finds locally sinusoidal perturbation, the property well known to everybody who has observed waves on water surface. We often see periodic water waves and rarely hear pure tones because surface waves are dispersive while sound waves are not. Surface waves form the main subject of analysis in this section but the ideas and results apply equally well to numerous other dispersive waves that exist in bulk fluids, plasma and solids. Dispersion usually results from some anisotropy or inhomogeneity of the medium. We shall try to keep our description universal when we turn to a consideration of non-linear dispersive waves having finite amplitudes. We shall consider weak non-linearity, assuming amplitudes to be small, and weak dispersion, which is possible in two distinct cases: (i) when the dispersion relation is close to acoustic and (ii) when waves are excited in a narrow spectral interval. These two cases correspond respectively to the Korteweg–de-Vries equation and the non-linear Schrödinger equation, which are as universal for dispersive waves as the Burgers equation for non-dispersive waves. In particular, the results of this chapter are as applicable to non-linear optics and quantum physics as to fluid mechanics. We have seen that without dispersion nonlinearity leads to breaking of sound waves and creation of shocks. Here we shall see that nonlinear steepening and dispersive spreading can
balance each other to create stationary nonlinear waves, in particular, soliton, quintessentially nonlinear object.

3.1 Linear waves

To have waves, one either needs inhomogeneity of fluid properties or compressibility. Here we consider an incompressible fluid with an extreme form of inhomogeneity – an open surface. In this section, we shall consider surface waves as an example of dispersive wave systems, account for gravity and surface tension as restoring forces and account for viscous friction. We then introduce general notions of phase and group velocities, which are generally different for dispersive waves. We discuss physical phenomena that appear because of that difference.

Linear waves in an infinite medium can be presented as superpositions of plane waves $\exp(ik\mathbf{r} - i\omega t)$. Therefore, linear waves of every type are completely characterized by the so-called dispersion relation between wave frequency $\omega$ and wavelength $\lambda$. As befits physicists and engineers, before making formal derivations we show simple ways to estimate $\omega(\lambda)$ up to a numerical factor. The simplest way is usually dimensional analysis. If gravity is a restoring force then the only relation between $\omega, \lambda, g$ is $\omega^2 \simeq g\lambda^{-1}$. If surface tension dominates, the frequency must depend on the coefficient of surface tension $\alpha$ (having dimensionality force/length=$\text{gram/s}^2$) as well fluid density $\rho$ which characterizes inertia. In this case, we have four parameters, $\omega, \lambda, \alpha, \rho$, and three dimensionalities, gram, centimetre and second, so that according to the $\pi$-theorem from Sect. ?? the expression for the frequency is unique again: $\omega^2 \simeq \alpha\lambda^{-3}\rho^{-1}$ up to a dimensionless factor.

If, however, gravity and surface tension are comparable, then five parameters and three dimensionalities does not allow one to determine the dispersion relation from dimensional analysis. One then ascends to a bit higher (yet still elementary) level of making an estimate by using Newton’s second law or its equivalent for small oscillations, called the virial theorem, which states that mean kinetic energy is equal to the mean potential energy.

Consider vertical oscillations of a fluid with the elevation amplitude $a$ and the frequency $\omega$. The fluid velocity can be estimated as $\omega a$ and the acceleration as $\omega^2 a$. When the fluid depth is much larger than the wavelength, we may assume
that the fluid layer of the order \( \lambda \) is involved in the motion. Newton’s second law for a unit area then requires that the mass \( \rho \lambda \) times the acceleration \( \omega^2 a \) must be equal to the gravitational force \( \rho \lambda a g \):

\[
\rho \omega^2 a \lambda \simeq \rho a g \Rightarrow \omega^2 \simeq g \lambda^{-1}.
\]  (3.1)

The same result one obtains using the virial theorem. Indeed, the kinetic energy per unit area of the surface can be estimated as the mass in motion \( \rho a \) times the velocity squared \( \omega^2 a^2 \). The gravitational potential energy per unit area can be estimated as the elevated mass \( \rho a \) times \( g \) times the elevation \( a \).

A curved surface has extra potential energy, whose density per unit area is the product of the coefficient of surface tension \( \alpha \) and the surface curvature \((a/\lambda)^2\). Taking potential energy as a sum of gravitational and capillary contributions, we obtain the dispersion relation for gravitational-capillary waves on deep water:

\[
\omega^2 \simeq g \lambda^{-1} + \alpha \lambda^{-3} \rho^{-1}.
\]  (3.2)

When the depth \( h \) is much smaller than \( \lambda \), fluid mostly moves horizontally with the horizontal velocity exceeding the vertical velocity \( \omega a \) by the geometric factor \( h/\lambda \). This makes the kinetic energy \( \rho \omega^2 a^2 \lambda^{-2} h^{-1} \) while the potential energy does not change. The virial theorem then gives the dispersion relation for waves on shallow water:

\[
\omega^2 \simeq g h \lambda^{-2} + \alpha h \lambda^{-4} \rho^{-1}.
\]  (3.3)

### 3.1.1 Surface gravity waves

Let us now formally describe fluid motion in a surface wave. As argued in Section ??, small-amplitude oscillations are irrotational flows. Then one can introduce the velocity potential that satisfies the Laplace equation \( \Delta \phi = 0 \) for incompressible flows. The pressure is

\[
p = -\rho (\partial \phi / \partial t + gz + v^2/2) \approx -\rho (\partial \phi / \partial t + gz),
\]

neglecting quadratic terms because the amplitude is small. As in considering Kelvin–Helmholtz instability in Section ?? we describe the surface form by the elevation \( \zeta(x,t) \). We can include the atmospheric pressure on the surface into \( \phi \rightarrow \phi + p_0 t / \rho \), which does not change the velocity field. We then have on the surface

\[
\frac{\partial \zeta}{\partial t} = \nu_z = \frac{\partial \phi}{\partial z}, \quad g \zeta + \frac{\partial \phi}{\partial t} = 0 \Rightarrow g \frac{\partial \phi}{\partial z} + \frac{\partial^2 \phi}{\partial t^2} = 0.
\]  (3.4)
The first equation here is the linearized kinematic boundary condition \( \frac{\partial \nu}{\partial t} = -g \frac{\partial \zeta}{\partial x} \), which states that the vertical velocity on the surface is equal to the time derivative of the surface height. The second one is the linearized dynamic boundary condition on the pressure being constant on the surface; it can also be obtained writing the equation for the horizontal acceleration due to gravity acting on an inclined surface: \( \frac{\partial \nu}{\partial t} = -g \frac{\partial \zeta}{\partial x} \). To solve (3.4) together with \( \Delta \phi = 0 \), one needs the boundary condition at the bottom: \( \frac{\partial \phi}{\partial z} = 0 \) at \( z = -h \). The solution of the Laplace equation periodic horizontally is exponential vertically:

\[
\phi(x, z, t) = a \cos(k x - \omega t) \cosh(z + h),
\]

\[
\omega^2 = gk \tanh kh.
\]

Differentiating the potential with respect to time and coordinates, we obtain

\[
\zeta = -a(\omega/g) \sin(k x - \omega t) \cosh(z + h),
\]

\[
\nu_x = -ak \sin(k x - \omega t) \cosh(z + h),
\]

\[
\nu_z = ak \cos(k x - \omega t) \sinh(z + h).
\]

Note that in the linear approximation \( \nu_x = gk \zeta / \omega \), i.e. fluid moves forward near crests and backward near troughs, as is known to every swimmer. The condition of weak non-linearity is \( \partial \nu / \partial t \gg \nu \partial \nu / \partial x \), which requires \( \omega \gg ak^2 = k v \) which can be written as \( g \gg k v^2 \).

The trajectories of fluid particles can be obtained by integrating the equation \( \dot{r} = \nu \) assuming small oscillations near some \( r_0 = (x_0, z_0) \), as for solving (3.4). Fluid displacement during the period can be estimated as velocity \( ka \) times \( 2\pi / \omega \); this is supposed to be much smaller than the wavelength \( 2\pi / k \). At first order in the small parameter \( ak^2 / \omega \), we find

\[
x = x_0 - \frac{ak}{\omega} \cos(k x_0 - \omega t) \cosh(z_0 + h),
\]

\[
z = z_0 - \frac{ak}{\omega} \sin(k x_0 - \omega t) \sinh(z_0 + h).
\]

The trajectories are ellipses described by

\[
\left( \frac{x - x_0}{\cosh(z_0 + h)} \right)^2 + \left( \frac{z - z_0}{\sinh(z_0 + h)} \right)^2 = \left( \frac{ak}{\omega} \right)^2.
\]

We see that as one goes down away from the surface, the amplitude of the oscillations decreases and the ellipses become more elongated as one approaches the bottom.
3 Dispersive waves

Figure 3.1 White particles suspended in the water are photographed during one period. The top figure shows a standing wave, where the particle trajectories are streamlines. The bottom figure shows a wave propagating to the right, some open loops indicate a Stokes drift to the right near the surface and compensating reflux to the left near the bottom. In both cases, the wave amplitude is 4% and the depth is 22% of the wavelength. From A. Wallett and F. Ruellan, *La Houille Blanche*, 5, 483–489 (1950).

One can distinguish two limits, depending on the ratio of the water depth to the wavelength. On shallow water \((kh \ll 1)\), the oscillations are almost one-dimensional: \(v_z/v_x \propto kh\). The dispersion relation is sound-like: \(\omega = \sqrt{ghk}\) (this formula is all one needs to answer the question in Exercise 3.1).

For gravity waves on deep water \((kh \gg 1)\), the frequency, \(\omega = \sqrt{gk}\), is like the formula for the period of the pendulum \(T = 2\pi/\omega = 2\pi\sqrt{L/g}\). Indeed, a standing surface wave is like a pendulum made of water, as seen in the top panel of Figure 3.1. For a running wave we have

\[
\begin{align*}
\xi &= -\frac{u}{\omega} \sin(kx - \omega t)e^{kz}, \\
v_x &= -u \sin(kx - \omega t)e^{kz}, \\
v_z &= u \cos(kx - \omega t)e^{kz}.
\end{align*}
\]

The fluid particles move in perfect circles whose radius exponentially decays with depth, with the rate equal to the horizontal wavenumber. This is again the property of the Laplace equation mentioned in Section ??: if the solution oscillates in one direction, it decays exponentially in the transverse direction. This supports our assumption that for a deep fluid, the layer comparable to wavelength is involved in motion, a fact known to divers.
3.1 Linear waves

At second order, we expect nonzero momentum and the mean (Stokes) drift. The reason is that the velocity potential must be purely periodic for a deep water, because the only other solution of the Laplace equation, \( x f(z) \), is ruled out by the requirement \( df(z)/dz \rightarrow 0 \) as \( z \rightarrow -\infty \). Since velocity cannot be \( x \)-independent, then counterflow to the Stokes drift is impossible, contrary to the case of sound discussed at the end of Section ???. Therefore, propagation of potential gravity waves on a deep water is always accompanied by a surface current.\(^1\) Since the total momentum of any horizontal level inside the fluid is zero for a periodic potential, the only contribution is from surface disturbance. The nonzero momentum appears in the second order as a product of \( v, \zeta \) disturbances of the first order. Multiplying (3.11) by (3.12) and averaging over the period we obtain the mean momentum density per unit area: \( \rho \langle \zeta v_x \rangle = \rho u^2/2\omega \). Exactly like in (??), we can also obtain the drift velocity:

\[
\langle v_x \rangle = -ku ((x-x_0) \cos(kx-\omega t) + (z-z_0) \sin(kx-\omega t)) = \frac{ku^2}{\omega}.
\]

From the general expression (??) for the displacement, one can also obtain the drift velocity as an average of \( k|\nabla \phi|^2/\omega = ku^2/\omega \) over the period. For small-amplitude wave, velocity is constant on a particle orbit and we obtain (3.14). In a deep tank of a finite length we expect the surface drift to be balanced by the counterflow at the bottom, as seen in the bottom panel of Figure 3.1. When the water is not deep one cannot disentangle surface and bottom currents; shallow-water waves can have zero mass flow, exactly like sound.

### 3.1.2 Viscous dissipation

The moment we account for viscosity, our solution (3.5) does not satisfy the boundary condition on the free surface,

\[
\sigma_{ij}n_j = \sigma_{ij}'n_j - pn_i = 0
\]
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(see Section ??), because both the tangential stress $\sigma_{xz} = -2\eta\phi_{xz}$ and the oscillating part of the normal stress $\sigma_{zz} = -2\eta\phi_{zz}$ are non-zero. Note also that (3.12) gives $v_i$ non-zero on the bottom. A true viscous solution has to be rotational but when viscosity is small, vorticity appears only in narrow boundary layers near the surface and the bottom. A standard derivation $^2$ of the rate of a weak decay is to calculate the viscous stresses from the solution (3.5) and substitute them into (??):

$$\frac{dE}{dt} = -\frac{\eta}{2} \int \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} \right)^2 dV = -2\eta \int \left( \frac{\partial^2 \phi}{\partial x_k \partial x_i} \right)^2 dV$$

$$= -2\eta \int \left( \phi_{xz}^2 + \phi_{zz}^2 + 2\phi_{xz}^2 \right) dV = -8\eta \int \phi_{xz}^2 dV.$$  

This means neglecting the narrow viscous boundary layers near the surface and the bottom (where there is not much motion by virtue of $kh \gg 1$). Assuming the decay to be weak (this requires $\nu k^2 \ll \omega$, which also guarantees that the boundary layer width $\sqrt{\nu/\omega}$ is smaller than the wavelength), we consider the energy averaged over the period, which is twice the averaged kinetic energy for small oscillations by the virial theorem:

$$\bar{E} = \frac{1}{2\pi/\omega} \int_0^{2\pi/\omega} E \, dt = \rho \int \overline{v^2} \, dV = 2\rho k^2 \int \overline{\phi^2} \, dV.$$  

The dissipation averaged over the period is related to the average energy:

$$\frac{\overline{dE}}{dr} = \frac{\partial \bar{E}}{\partial r} \omega/2\pi = -4\nu k^4 \int \overline{\phi^2} \, dV = -4\nu k^2 \bar{E}. \quad (3.15)$$

If you are not confused, you are not paying attention. Indeed, something seems strange in this derivation. Notice that our solution (3.5) satisfies the Navier–Stokes equation (but not the boundary conditions) since the viscous term is zero for the potential flow: $\Delta \mathbf{v} = 0$. How then can zero viscous force give non-zero dissipation? To answer that, note that the force $\rho v \Delta \mathbf{v} = \partial \sigma_{ik}/\partial x_k$ is the viscous stress divergence, which is zero. But the stress $\sigma_{ik}$ itself is non-zero. In other words, the net viscous force on any fluid element is zero, but the viscous forces around an element are non-zero as it deforms (compare with Exercise 1.16). These forces bring the energy dissipation, which is $\sigma_{ik} \partial v_i/\partial x_k$. Indeed, we have shown in (??) that the viscous contribution to the energy change consists of two terms: $v_i \partial \sigma_{ik}/\partial x_k = \partial v_i \sigma_{ik}/\partial x_k - \sigma_{ik} \partial v_i /\partial x_k$. The first term is the divergence and describes the transport of energy, while the second term describes dissipation. This second term has a non-zero time average...
so that viscosity causes waves to decay. Moreover, $\sigma'_k \partial v_i / \partial x_k$ is distributed over the fluid bulk rather than being concentrated in the viscous boundary layer, in distinction from the term $v_i \sigma'_k / \partial x_k = \mathbf{v} \Delta \mathbf{v}$. To appreciate how the bulk integration can present the boundary layer distortion, consider a function $U(x)$ on $x \in [0, 1]$, which is almost linear but curves in a narrow vicinity near $x = 1$ to give $U'(1) = 0$, as shown in Figure 3.2. Then the integral $\int_0^1 U U'' \, dx = -\int_0^1 (U')^2 \, dx$ is non-zero, and the main contribution is from the bulk.

However, the first derivation of the viscous dissipation (3.15) was made by Stokes in quite an ingenious way, which did not involve any boundary layers. Since the potential solution satisfies the Navier–Stokes equation, he suggested *imagining* how one may also satisfy the boundary conditions (so that no boundary layers appear). First, we need an extensible bottom to move with $v_x(-h) = -k \sin(kx - \omega t)$. Since $\sigma_{zz}(0) = 0$, such bottom movements do no work. In addition, we must apply extra forces to the fluid surface to compensate for $\sigma_{zz}(0)$ and $\sigma_{xz}(0)$. Such forces do work (per unit area per unit time) $v_x \sigma_{zz} + v_z \sigma_{xz}$. After averaging over the period of the monochromatic wave, this becomes $4 \eta k^2 \phi \phi_z$, which is $8 \nu k^2$ times the average kinetic energy per unit surface area, $\rho \phi \phi_z/2$ — to obtain this, one writes

$$\rho \int (\nabla \phi)^2 \, dV = \rho \int \nabla \cdot (\phi \nabla \phi) \, dV = \int \phi \phi_z \, dS.$$  

Since we have introduced forces that make our solution steady (it satisfies equations and boundary conditions), the work of those forces exactly equals the rate of the bulk viscous dissipation which is thus $4 \nu k^2$ times the total energy.

### 3.1.3 Capillary waves

Surface tension creates an extra pressure difference proportional to the curvature of the surface:

![Figure 3.2](image)

Figure 3.2. A function whose $U''$ is small everywhere except a small vicinity of $x = 1$ yet $\int_0^1 U U'' \, dx = -\int_0^1 (U')^2 \, dx$ is due to the bulk.
3 Dispersive waves

This changes the second equation of (3.4),

\[ g \zeta + \frac{\partial \phi}{\partial t} = \frac{\alpha \rho}{\partial x^2} \zeta, \]  

(3.16)

and adds an extra term to the dispersion relation

\[ \omega^2 = \left( gk + \alpha k^3 / \rho \right) \tanh kh. \]  

(3.17)

That two restoring factors enter additively into \( \omega^2 \) is due to the virial theorem, as explained at the beginning of Section 3.1.

We see that there is a wavenumber \( k_* = \sqrt{\rho g / \alpha} \), which separates gravity-dominated long waves from short waves dominated by surface tension. In general, both gravity and surface tension provide a restoring force for the surface perturbations. For water, \( \alpha \simeq 70 \text{ erg cm}^{-2} \) and \( \lambda_* = 2\pi / k_* \simeq 1.6 \text{ cm} \). Now we can answer why water cannot be held in an upside-down glass. The question seems bizarre only to a non-physicist, since physicists usually know that atmospheric pressure at normal conditions is of the order of \( P_0 \simeq 10^5 \text{ newtons per square metre} \), which is enough to support up to \( P_0 / \rho g \simeq 10 \text{ metres of water column} \). That is, if the fluid surface remained plane then the atmospheric pressure would keep the water from spilling. The relation (3.17) tells us that with a negative gravity, the plane surface is unstable:

\[ \omega^2 \propto -gk + \alpha k^3 < 0 \text{ for } k < k_*. \]

On the contrary, water can be kept in an upside-down capillary with a diameter smaller than \( \lambda_* \) since the unstable modes cannot fit in (see Exercise 3.10 for the consideration of a more general case).

3.1.4 Phase and group velocity

Let us discuss now general properties of one-dimensional propagation of linear dispersive waves. To describe the propagation, we use the Fourier representation, since every harmonic \( \exp(ikx - i\omega t) \) propagates with the constant velocity \( \omega_k / k \), completely determined by the frequency–wavenumber relation \( \omega_k \). One
3.1 Linear waves

can learn quite general lessons by considering the perturbation in the simple Gaussian form,

$$\zeta(x, 0) = \zeta_0 \exp\left(ik_0x - x^2/l^2\right),$$

for which the distribution in $k$-space is Gaussian as well:

$$\zeta(k, 0) = \int dx \zeta_0 \exp\left[i(k_0 - k)x - x^2/l^2\right] = \zeta_0 l\sqrt{\pi} \exp\left[-\frac{l^2}{4}(k_0 - k)^2\right].$$

This distribution has the width $1/l$. Consider first a spatially wide and spectrally narrow wave packet (quasi-monochromatic wave) with $k_0 l \gg 1$ so that we can expand

$$\omega(k) = \omega_0 + (k - k_0)\omega' + (k - k_0)^2\omega''/2 \quad (3.18)$$

and substitute it into

$$\zeta(x, t) = \int \frac{dk}{2\pi} \zeta(k, 0) \exp ikx - i\omega_0 t \quad (3.19)$$

$$\approx \frac{\zeta_0}{2} e^{i(k_0x - i\omega_0 t)} \exp\left[-\frac{(x - \omega t)^2}{(l^2 + 2it\omega'')/4}\right]^{-1/2}. \quad (3.20)$$

We see that the perturbation $\zeta(x, t) = \exp(ik_0x - i\omega_0 t)\Psi(x, t)$ is a monochromatic wave with a complex envelope having the modulus

$$|\Psi(x, t)| \approx \frac{\zeta_0}{L(t)} \frac{l}{\omega''} \exp\left[-(x - \omega t)^2/L^2(t)\right],$$

$$L(t) = t^4 + (t\omega'')^{21/4}. \quad (3.21)$$

The phase propagates with the phase velocity $\omega_0/k_0$, while the envelope (and the energy determined by $|\Psi|^2$) propagate with the group velocity $\omega'$. The maximum is at $x = \omega t$ where waves with close wavevectors interfere constructively, away from this point the waves cancel each other. For sound waves, $\omega_0 = ck$, the group and the phase velocities are equal and are the same for all wavenumbers; the wave packet does not spread, since $\omega'' = 0$. On the contrary, when $\omega'' \neq 0$, the waves are called dispersive, since different harmonics move with different velocities and disperse in space; the wave packet spreads with time and its amplitude decreases since $L(t)$ grows. For $\omega_0 \propto k^{\alpha}$, $\omega' = \alpha\omega_0/k$. In particular, the group velocity for gravity waves on deep water is half that of the phase velocity so that individual crests can be seen appearing out of nowhere at the back of the packet and disappearing at the front. For capillary waves on deep water, the group velocity is 1.5 times more, so crests appear at the front.
Consider now a spatially localized initial perturbation, which corresponds to a wide distribution in $k$-space with many harmonics coordinating their phases in such a way as to provide constructive interference inside a narrow region and destructive interference outside. For dispersive waves ($\omega'' \neq 0$) different harmonics propagate with different velocities and separate. After a long time, we shall see periodic perturbation with the wavelength depending on the position.

Indeed, in the integral (3.19) for given large $x, t$ the main contribution is given by the wavenumber determined by the stationary phase condition $\omega'(k_c) = x/t$, i.e. waves of wavenumber $k_c(x, t)$ are found at positions moving forward with the group velocity $\omega'(k_c)$. The spectral form of the perturbation is irrelevant in this limit and can be considered constant. Substituting $\zeta(k, 0) = 1$ (or taking a limit $l \rightarrow 0$, $\zeta_0 l \sqrt{\pi} = 1$) we obtain from (3.19)

$$\zeta(x, t) \approx \frac{1}{2} e^{ik_c x - i\omega(k_c)t} \left[ it\omega''(k_c)/2 \right]^{-1/2}. \tag{3.22}$$

The dependence of the wave envelope on $x, t$ is thus determined by the factor $t\omega''(k_c)^{-1/2}$. Water waves have non-monotonic dependence of the group velocity on the wavenumber so that the equation $\omega'(k_c) = x/t$ has two solutions. Smaller wavenumber corresponds to a gravity wave and larger wavenumber to a capillary wave; these waves then can propagate together, see Exercise 3.3.

For gravity waves, $\omega'(k_c) = \sqrt{g/4k_c} = x/t$ gives $k_c = gt^2/4x^2$ and the envelope behaves as $|\zeta(x, t)| \propto \sqrt{gt^2/x^4}$. It decays with distance at a given time and grows with time at a given point. For capillary waves, $\omega'(k_c) = (3/2) \sqrt{\alpha k_c/\rho} = x/t$ gives $k_c = 4\rho x^2/9\alpha t^2$ and $|\zeta(x, t)| \propto x^{1/2}/t$. Counter-intuitively, we see that the amplitude of the capillary wave train actually grows with the distance. That growth is restricted by the condition $k_c l \ll 1$ which restricts the distance by $x \ll \sqrt{\alpha/\rho l}$. At larger distances, the amplitude decays with the distance; in reality, short capillary waves are effectively attenuated by viscosity.

The consideration leading to (3.22) apparently breaks down as we approach the point where $k_c(x, t) = k_*, \text{ such that } \omega''(k_*) = 0$, that is where the group velocity has maximum or minimum. That point is caustic since it is the boundary
between a region with no waves and a region with two group of waves. Group velocity is stationary on caustics so that waves with close wavevectors run together. Describing the long-time behavior at the vicinity of the caustic requires further expansion of \( \omega(k) \) up to cubic terms. Assuming \( \omega'(k_x) - x/t \) small but nonzero, denoting \( \omega(k_x) = \omega_0 \) and \( \omega'(k_x) = v, \) we obtain

\[
\xi(x,t) \approx e^{ik_x x - i\omega_0 t} \int \frac{dk}{2\pi} \exp(i(k-k_0)(x-v_s t) - i(k-k_0)^2 \omega''(k_0)t/6) \tag{3.23}
\]

Here we denoted \( X = (v_s t - x)/\omega''(k_0)t/2^{1/3} \) and introduced the Airy integral \( Ai(X) \) playing near caustic the same role as the Gaussian integral (3.20) at a general point. The Airy integral as a function of \( X \) is shown in the Figure, which can be understood, again, with the help of the stationary phase approach. Far behind the caustic where \( X > 1 \), the integral is determined by the imaginary saddle point \( s = i\sqrt{X} \) and is exponentially small: \( Ai(X) \propto \exp(-2X^{3/2}/3) \).

In front of the caustics where \( X \) is negative, we have two real saddle points with \( s = \pm \sqrt{-X} \) and \( Ai(X) \propto |X|^{-1/4} \cos(2X^{3/2}/3 - \pi/4) \). The wave amplitude reaches its maximum slightly ahead of the caustic. Note that the amplitude around the caustic decays as \( t^{-1/3} \), that is slower than the decay \( t^{-1/2} \) described by (3.21) for a plane quasi-monochromatic wave with a generic wavenumber.

In a \( d \)-dimensional case, the approach (3.20) based on the Gaussian integral gives instead of (3.22) the following expression

\[
\xi(x,t) = e^{ik_x x - i\omega_0(k_0)t} \int \frac{dk_1 \ldots dk_d}{2\pi} \exp(-itk_x k_j \partial^2 \omega/\partial k_i \partial k_j/2) \approx e^{ik_x x - i\omega_0(k_0)t} (\deti \partial^2 \omega/\partial k_i \partial k_j/2\pi)^{-1/2}. \tag{3.24}
\]

The caustic appears for such \( x,t \) where the determinant of the matrix \( \partial^2 \omega/\partial k_i \partial k_j \) vanishes at \( k_0 \) defined by \( \partial \omega(k_0)/\partial k = x/t \). Sufficient condition for that is vanishing of a single eigenvalue of the matrix. The direction of the respective eigenvector defines the normal to the caustic, which is a \( (d-1) \)-dimensional surface in \( x \)-space. We then have Airy integral along this direction and a usual Gaussian integration along remaining \( d-1 \) directions, which gives the amplitude on the caustic decaying as \( t^{-1/3-(d-1)/2} \).

### 3.1.5 Wave generation

An obstacle to the stream or wave source moving with respect to water can generate a steady wave pattern if the projection of the source’s relative velocity
$V$ in the direction of wave propagation is equal to the phase velocity $c(k) = \omega(k)/k$. For example, if the source creates an elevation of the water surface then it must stay on the wave crest, which moves with the phase velocity. If the wave propagates at the angle $\theta$ to the direction of the source motion, then the condition $V \cos \theta = c$ for generation of a stationary wave pattern is a direct analogue of the resonance condition for Vavilov–Cherenkov radiation by particles moving faster than light in a medium. Note also similarity to the Landau criterion for superfluidity: In a fluid moving with the velocity $v$, an obstacle or a wall can generate excitation with the momentum $p$ and the energy $\epsilon(p)$ if the resulting energy change is negative, $\epsilon(p) + (p \cdot v) < 0$, which requires $v > \epsilon(p)/p$. If the spectrum of excitations has a minimal $\epsilon(p)/p$ then the fluid moving with $v < \text{min} \epsilon(p)/p$ meets no resistance. For gravity-capillary surface waves on deep water, the requirement for the Vavilov–Cherenkov resonance means that $V$ must exceed the minimal phase velocity, which is

$$c(k_0) = \frac{\omega(k_0)}{k_0} = \left(\frac{4\alpha g \rho}{\rho}\right)^{1/4} \simeq 23 \text{ cm s}^{-1}. \quad (3.25)$$

When the phase velocity is minimal it coincides with the group velocity:

$$\frac{\partial \omega}{\partial k} \frac{\omega}{k} = \frac{1}{k} \left(\frac{\partial \omega}{\partial k} - \frac{\omega}{k}\right) = 0.$$ 

Let us stress that the minimal group velocity $v_*$ is smaller than $c(k_0)$ and corresponds to a larger wavelength, $\lambda_* > \lambda_0$, see Figure. Minimal phase velocity determines the threshold speed for generating waves while minimal group velocity determines the speed of the caustic.
Consider first a source that is long in the direction perpendicular to the motion, say a tree fallen across the stream. Then a one-dimensional pattern of surface waves is generated with two wavenumbers that correspond to $c(k) = V$. These two wavenumbers correspond to the same phase velocity but different group velocities. The waves transfer energy from the source with the group velocity. Since the longer (gravity) wave has its group velocity lower, $\omega' < V$, then it is found behind the source (downstream), while the shorter (capillary) wave can be found upstream. Of course, capillary waves can only be generated by a really thin object (much smaller than $\lambda_*$), like a fishing line or a small tree branch.

Generally, consider a medium where a wave with the wavenumber $k$ has the frequency $\omega(k)$ and a small decay rate $\gamma_k$. Assume that the wave source generating spectral density $A(k)$ started moving with the speed $V$ relative to the medium at $x \to -\infty$ at time $t \to -\infty$ and is at $x = 0$ at $t = 0$. At that time the perturbation at every point $x$ is the sum of the plane waves generated during the past:

$$
\int_{-\infty}^{\infty} A(k) \, dk \int_{-\infty}^{0} dx' \int_{-\infty}^{0} dt' \delta(t' + x'/V) e^{i k(x-x') - i k x'/V}
$$

$$
= \int_{-\infty}^{\infty} A(k) \, dk \int_{-\infty}^{0} dx' e^{i k(x-x') + (i \omega + i \gamma_k) x'/V} = V \int \frac{i e^{ikx} \, dk}{k V - \omega(k) + i \gamma_k}.
$$

(3.26)
Dispersive waves

Since $\gamma k \ll \omega(k)$, the main contribution into the integral is given by $k$ close to $k_0 = \omega(k_0)/V$, which corresponds to the wave whose phase velocity coincides with the speed of the source. Then we can expand $\omega(k) = \omega(k_0) + \omega'(k_0)\Delta k$ and write (3.26) as follows:

$$VA(k_0)e^{i\theta} \int_{-\infty}^{\infty} \frac{e^{ikx}d\Delta k}{w\Delta k + i\gamma_0}.$$ (3.27)

Here $\gamma_0 = \gamma(k_0)$ and $w = V - \omega'(k_0) = \omega(k_0)/k_0 - \omega'(k_0)$ is the difference between the phase and group velocities. We also assumed that the source spectral density does not change when $k$ changes by the values of the order $\gamma/w$, that is that the source size is less than $w/\gamma$. It is convenient to compute the integral (3.27) making the path a closed contour in the complex plane of $\Delta k$ by adding a semi-circle at infinity where the integrand is infinitesimal. For positive/negative $x$ such semi-circle is in the upper/lower half plane. The integrand has a pole at $\Delta k = -i\gamma_0/w$. When $w > 0$ the pole is in the lower half plane of complex $\Delta k$, so that the integral is zero for $x > 0$, while for $x < 0$ it is

$$\frac{2\pi V}{w} \exp(ik_0x + \gamma_0x/w).$$ (3.28)

Indeed, when the phase velocity is larger than the group velocity the perturbation is behind the source. The same formula (3.28) describes the case $w < 0$ and $x > 0$. The temporal decay rate $\gamma_0$ determines how the perturbation decays away from the source in space.

Figure 3.3 Pattern of waves generated by a ship that moves along the $x$-axis to the left. The circle is the locus of points reached by waves generated at A when the ship arrives at B. The broken lines show the Kelvin wedge. The solid lines are wave crests.
Ships generate an interesting pattern of gravity waves on the water surface that can be understood as follows. The wave generated at the angle $\theta$ to the ship’s motion has its wavelength determined by the condition $V \cos \theta = c(k)$, necessary for the ship’s bow to stay on the wave crest. This condition means that different wavelengths are generated at different angles in the interval $0 \leq \theta \leq \pi/2$. Similar to our consideration in Section ??, let us find the locus of points reached by the waves generated at A during the time it takes for the ship to reach B, see Figure 3.3. The waves propagate away from the source with group velocity $\omega' = c(k)/2$. The fastest is the wave generated in the direction of the ship propagation ($\theta = 0$) which moves with group velocity equal to half the ship’s speed and reaches $A'$ such that $AA' = AB/2$. The wave generated at the angle $\theta$ reaches E such that $AE = AA' \cos \theta$ which means that $AEA'$ is a right angle. We conclude that the waves generated at different angles reach the circle with the diameter $AA'$. Since $OB = 3OC$, all the waves generated before the ship reached B are within the Kelvin wedge with the angle $\varphi = \arcsin(1/3) \approx 19.5^\circ$; compare this with the Mach cone shown in Figure ?? . Note the remarkable fact that the angle of the Kelvin wedge is completely universal, i.e. independent of the ship’s speed.

Let us describe now the form of the wave crests, which are neither straight nor parallel since they are produced by the waves emitted at different moments at different angles. The wave propagating at the angle $\theta$ makes a crest at the

Figure 3.4 The wave pattern of a ship consists of the breaking wave from the bow with its turbulent wake (distinguished by a short trace of white foam) and the Kelvin pattern. Inside the Kelvin wedge one can see the line of maxima created by wavelengths comparable to the ship length. Photograph copyright: Alexey Baskakov, www.dreamstime.com.
Dispersive waves

Consider point E on a crest with the coordinates

\[ x = AB \left(2 - \cos^2 \theta\right)/2, \quad y = (AB/2) \sin \theta \cos \theta. \]

The crest slope must satisfy the equation \( \frac{dy}{dx} = \cot \theta \), which gives \( \frac{dAB}{d\theta} = -AB \tan \theta \). The solution of this equation, \( AB = X_1 \cos \theta \), describes how the source point A is related to the angle \( \theta \) of the wave that creates the given crest. Different integration constants \( X_1 \) correspond to different crests. The crest shape is given parametrically by

\[ x = X_1 \cos \theta \left(2 - \cos^2 \theta\right)/2, \quad y = \left(X_1/4\right) \cos \theta \sin 2\theta \quad (3.29) \]

and it is shown in Figure 3.3 by the solid lines, see also Figure 3.4. As expected, longer (faster) waves propagate at smaller angles. Note that for every point inside the Kelvin wedge one can find two different source points. That means that two constant-phase lines cross at every point, like the crossing of crests seen at the point F. One sees two distinct families of waves: diverging from the ship and transverse to the direction of motion. There are no waves outside the Kelvin wedge, whose boundary is thus a caustic, where every crest has a cusp like at the point D in Figure 3.3, determined by the condition that both \( x(\theta) \) and \( y(\theta) \) have maxima. Differentiating (3.29) and solving \( \frac{dx}{d\theta} = \frac{dy}{d\theta} = 0 \), we obtain the propagation angle \( \cos^2 \theta_0 = 2/3 \), which is actually the angle CAB since it corresponds to the wave that reached the wedge. We can express the angle COA alternatively as \( \pi/2 + \phi \) and \( \pi - 2\theta_0 \) and relate: \( \theta_0 = \pi/4 - \phi/2 \approx 35^\circ \).

### 3.2 Weakly non-linear waves

The law of linear wave propagation is completely characterized by the dispersion relation \( \omega_k \). It does not matter what physical quantity oscillates in the wave (fluid velocity, density, electromagnetic field, surface elevation, etc.), waves with the same dispersion relation propagate in the same way. Can we achieve the same level of universality in describing non-linear waves? As we shall see in this and the next sections, some universality classes can be distinguished but the level of universality naturally decreases as non-linearity increases.

#### 3.2.1 Hamiltonian description

What else, apart from \( \omega_k \), must we know to describe weakly non-linear waves? Since every wave is determined by two dynamic variables, amplitude and phase,
it is natural to employ the Hamiltonian formalism where variables also come in pairs (coordinate-momentum, action-angle). That must work for waves in a conservative medium. Indeed, the Hamiltonian formalism is the most general way to describe systems that satisfy the least-action principle, as most closed physical systems do. The main advantage of the Hamiltonian formalism (compared, say, with its particular case, the Lagrangian formalism) is an ability to use canonical transformations. Those transformations involve both coordinates and momenta and are thus more general than the coordinate transformations one uses within the Lagrangian formalism, which employs coordinates and their time derivatives (do not confuse Lagrangian formalism in mechanics and field theory with the Lagrangian description in fluid mechanics). Canonical transformation is a powerful tool that allows one to reduce a variety of problems into a few universal problems. So let us try to understand what is a general form of the Hamiltonian of a weakly non-linear wave system.

As we have seen in Section ??, Hamiltonian mechanics of continuous systems lives in an even-dimensional space of coordinates \( q(r, t) \) and momenta \( \pi(r, t) \) that satisfy the equations

\[
\frac{\partial q(r, t)}{\partial t} = \frac{\delta H}{\delta \pi(r, t)}, \quad \frac{\partial \pi(r, t)}{\partial t} = -\frac{\delta H}{\delta q(r, t)}.
\]

Here the Hamiltonian \( H(q(r, t), \pi(r, t)) \) is a functional (simply speaking, a function presents a number for every number, while a functional presents a num-
ber for every function; for example, a definite integral is a functional. The vari-
ational derivative $\frac{\delta I}{\delta f(r)}$ is a generalization of the partial derivative $\frac{\partial}{\partial f(r_n)}$ from a discrete to a continuous set of variables. The variational derivative of a
linear functional of the form $I[f] = \int \phi(r') f(r') \, \mathrm{d}r'$ is calculated by

$$
\frac{\delta I}{\delta f(r)} = \int \phi(r') \frac{\delta f(r')}{\delta f(r)} \, \mathrm{d}r' = \int \phi(r') \delta(r-r') \, \mathrm{d}r' = \phi(r).
$$

For this, one mentally replaces $\frac{\delta}{\delta f(r)}$ with $\frac{\partial}{\partial f(r_n)}$ and integration with
summation.

For example, the Euler and continuity equations for potential flows (in
particular, acoustic waves) can be written in a Hamiltonian form:

$$
\frac{\partial \rho}{\partial t} = \frac{\delta H}{\delta \phi}, \quad \frac{\partial \phi}{\partial t} = -\frac{\delta H}{\delta \rho},
$$

(3.30)

$$
\mathcal{H} = \int \rho \left[ \frac{|\nabla \phi|^2}{2} + E(\rho) \right] \, \mathrm{d}r.
$$

We shall use canonical variables even more symmetrical than $\pi, q$, analogous
to what are called creation–annihilation operators in quantum theory. In our
case, they are just functions, not operators. Assuming $p$ and $q$ to be of the same
dimensionality (which can always be achieved by multiplying them by factors)
we introduce

$$
a = \frac{q + i\pi}{\sqrt{2}}, \quad a^* = \frac{q - i\pi}{\sqrt{2}}.
$$

Instead of two real equations for $p, q$ we now have one complex equation

$$
\frac{1}{i} \frac{\partial a(r, t)}{\partial t} = \frac{\delta H}{\delta a^*(r, t)}.
$$

(3.31)

The complex conjugated equation describes the evolution of $a^*$.

In the linear approximation, waves with different wavevectors do not interact
and their equations are independent. Normal canonical coordinates in infinite
space are the complex Fourier amplitudes $a_k$, which satisfy the equation

$$
\frac{\partial a_k}{\partial t} = -i\omega_k a_k.
$$

Comparing this with (3.31) we conclude that the Hamiltonian of a linear wave
system is quadratic in the amplitudes:

$$
\mathcal{H}_2 = \int \omega_k |a_k|^2 \, \mathrm{d}k.
$$

(3.32)
Weakly non-linear waves

It is the energy density per unit volume. Terms of higher orders describe wave interaction due to non-linearity; the lowest terms are cubic:

$$H_3 = \int \left[ (V_{123} a_1^* a_2 a_3 + \text{c.c.}) \delta(k_1 - k_2 - k_3) 
+ (U_{123} a_1^* a_2^* a_3^* + \text{c.c.}) \delta(k_1 + k_2 + k_3) \right] d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3.$$ (3.33)

Here c.c. means the complex conjugated terms and we use shorthand notations $a_1 = a(k_1)$, etc. The delta functions express momentum conservation and appear because of space homogeneity. Indeed, (3.32) and (3.33) are, respectively, the Fourier representation of the integrals, like

$$\int \Omega(\mathbf{r}_1 - \mathbf{r}_2) a(\mathbf{r}_1) a(\mathbf{r}_2) d\mathbf{r}_1 d\mathbf{r}_2$$

and

$$\int V(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}_1 - \mathbf{r}_3) a(\mathbf{r}_1) a(\mathbf{r}_2) a(\mathbf{r}_3) d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}_3.$$

The Hamiltonian is real and its coefficients have obvious symmetries, $U_{123} = U_{132} = U_{213}$ and $V_{123} = V_{132}$. We presume that every next term in the Hamiltonian expansion is smaller than the previous one, in particular, $H_2 \gg H_3$, which requires

$$\omega_k \gg V|a_k|k^d, \quad U|a_k|k^d,$$ (3.34)

where $d$ is space dimensionality (two for surface waves and three for sound, for instance).

In the perverse nature of people who learnt quantum mechanics before fluid mechanics, we may use an analogy between $a, a^*$ and the quantum creation–annihilation operators $a, a^+$ and suggest that the $V$-term must describe the confluence $2 + 3 \rightarrow 1$ and the reverse process of decay $1 \rightarrow 2 + 3$. Similarly, the $U$-term must describe the creation of three waves from a vacuum and the opposite process of annihilation. We shall use quantum analogies quite often in this chapter since quantum physics is to a large extent a wave physics. To support this quantum-mechanical interpretation and make explicit the physical meaning of different terms in the Hamiltonian, we write a general equation of motion for weakly non-linear waves:

$$\frac{\partial a_k}{\partial t} = -3i \int U_{k12} a_1^* a_2^* \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}) d\mathbf{k}_1 d\mathbf{k}_2$$

$$- i \int V_{k12} a_1 a_2 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) d\mathbf{k}_1 d\mathbf{k}_2$$ (3.35)
where we have also included the linear damping $\gamma_k$ (as one always does when resonances are possible). The delta functions in the integrals suggest that each respective term describes the interaction between three different waves. To see this explicitly, consider a particular initial condition with two waves, having, respectively, wavevectors $k_1, k_2$, frequencies $\omega_1, \omega_2$ and finite amplitudes $A_1, A_2$. Then the last non-linear term in (3.35), $-2ie^{i(\omega_2 - \omega_1)t} \int k_2 A_1^2 \delta(k_1 - k_2 - k) \, dk_1 \, dk_2 - (\gamma_k + i\omega_k) a_k,$ provides the wave $k = k_1 - k_2$ with periodic forcing of frequency $\omega_1 - \omega_2$, and similarly the other two terms. The forced solution of (3.35) is then as follows:

$$a(k, t) = -3ie^{i(\omega_1 + \omega_2)t} \frac{U_{k_1 2} A_1^2 A_2^2 \delta(k_1 + k_2 + k)}{\gamma_k + i(\omega_1 + \omega_2 + \omega_k)} - ie^{-i(\omega_1 + \omega_2)t} \frac{V_{k_1 2} A_1 A_2^2 \delta(k_1 + k_2 - k)}{\gamma_k + i(\omega_1 + \omega_2 - \omega_k)} - 2ie^{i(\omega_2 - \omega_1)t} \frac{V_{k_1 2}^* A_1 A_2^2 \delta(k + k_2 - k_1)}{\gamma_k + i(\omega_1 + \omega_2 - \omega_1)}.$$ 

Here and below, $\omega_{1,2} = \omega(k_{1,2})$. Because of (3.34) the amplitudes of the secondary waves are small except for the cases of resonances, that is when the driving frequency coincides with the eigenfrequency of the wave with the respective $k$. The amplitude $a(k_1 + k_2)$ is not small if $\omega(k_1 + k_2 + \omega(k_1) + \omega(k_2)) = 0$ – this can happen in the non-equilibrium medium where negative-frequency waves are possible. Negative frequency corresponds to negative energy (3.32), which means that excitation of the wave decreases the energy of the medium. This may be the case, for instance, when there are currents in the medium and the wave moves against the current. In non-equilibrium medium, the frequency can also be complex (which signals instability), then $\mathcal{H}_2$ is different from (3.32) – see Exercise 3.4. Two other resonances require $\omega(k_1 + k_2) = \omega(k_1) + \omega(k_2)$ and $\omega(k_1 - k_2) = \omega(k_1) - \omega(k_2)$ – the dispersion relations that allow for this are called dispersion relations of the decay type. For example, the power dispersion relation $\omega_2 \propto k^\alpha$ is of the decay type if $\alpha \geq 1$ and of the non-decay type (that is, does not allow for the three-wave resonance) if $\alpha < 1$, see Exercise 3.5.

### 3.2.2 Hamiltonian normal forms

Intuitively, it is clear that non-resonant processes are unimportant for weak non-linearity. Technically, one can use the canonical transformations to eliminate
the non-resonant terms from the Hamiltonian. Because the terms that we want to eliminate are small, the transformation should be close to identical. Consider some continuous distribution of \( a_k \). If one wants to get rid of the \( U \)-term in \( H_3 \), one makes the following transformation

\[
b_k = a_k - 3 \int \frac{U_{k1k2} a_1^* a_2^*}{\omega_1 + \omega_2 + \omega_k} \delta(k_1 + k_2 + k) \, dk_1 \, dk_2.
\] (3.36)

It is possible when the denominator does not turn into zero in the integration domain determined by the delta-function of the wave vectors, that is when the spatial-temporal resonance is impossible. This is the case, in particular, for all media that were in thermal equilibrium before wave excitation. The Hamiltonian \( \mathcal{H}(b, b^*) = H_2 + H_3 \) does not contain the \( U \)-term. The elimination of the \( V \)-terms is made by a similar transformation

\[
b_k = a_k + \int \left[ \frac{V_{k1k2} a_1^* a_2^*}{\omega_1 + \omega_2 - \omega_k} \delta(k_1 + k_2 - k) \right.
\]

\[
+ \left. \frac{V_{k2k3} a_2^* a_3^*}{\omega_2 - \omega_1 - \omega_k} \delta(k_2 + k_3 - k) \right] \, dk_1 \, dk_2.
\] (3.37)

possible only for the non-decay dispersion relation. We see that both transformations (3.36, 3.37) are possible when denominators do not turn into zero, that is when the respective processes are non-resonant. One can check that the transformations are canonical, that is \( i\dot{b}_k = \frac{d}{db^*_k} \mathcal{H}(b, b^*) \). The procedure described here was invented for excluding non-resonant terms in celestial mechanics and later generalized for continuous systems.\(^8\)

We may thus conclude that (3.33) is the proper Hamiltonian of interaction only when all three-wave processes are resonant. When there are no negative-energy waves but the dispersion relation is of the decay type (like for capillary waves on deep water), the proper interaction Hamiltonian contains only the \( V \)-term. When the dispersion relation is of the non-decay type (like for gravity waves on water), all the cubic terms can be excluded and the interaction Hamiltonian must be of the fourth order in wave amplitudes. Moreover, if the dispersion relation does not allow \( \omega(k_1 + k_2 + k_3) = \omega(k_1) + \omega(k_2) + \omega(k_3) \) then it does not allow \( \omega(k_1 + k_2 + k_3) = \omega(k_1) + \omega(k_2) + \omega(k_3) \) as well. This means that when decays of the type \( 1 \rightarrow 2 + 3 \) are non-resonant, four-wave decays like \( 1 \rightarrow 2 + 3 + 4 \) are non-resonant too. So the proper Hamiltonian in this case describes four-wave scattering \( 1 + 2 \rightarrow 3 + 4 \), which is always resonant:

\[
\mathcal{H}_4 = \int T_{1234} a_1 a_2 a_3^* a_4^* \delta(k_1 + k_2 - k_3 - k_4) \, dk_1 \, dk_2 \, dk_3 \, dk_4.
\] (3.38)
One may ask: why bother with transformations (3.36,3.37) and not just omit non-resonant terms from the Hamiltonian? The point is that in the new variables the remaining interaction coefficients change. For example, after excluding cubic terms, $T_{1212}$ acquires additions of the type $|V|_1 V_1 + V_2^2 / (\omega_1 + \omega_2 - \omega_1 - \omega_2)$. If there are surfaces in the space $\{k_1, k_2\}$ where the denominator $\omega_1 + \omega_2 - \omega_1 - \omega_2$ is small (i.e., the cubic processes are almost resonant, as for sound waves with small positive dispersion), such additions could be dominant.

### 3.2.3 Wave instabilities

Wave motion itself can be unstable with respect to small perturbations. Let us show that if the dispersion relation is of the decay type, then a monochromatic wave of sufficiently high amplitude is subject to a decay instability. Consider the medium that contains a finite-amplitude wave $A \exp(i k r - \omega_k t)$ and add initial perturbations in the form of two waves with small amplitudes $a_1, a_2$. For interaction to have a net effect, all three waves must have wavevectors in a spatial resonance: $k_1 + k_2 = k$. We write linearized equations on perturbation leaving only resonant terms in (3.35):

$$
\dot{a}_1 + (\gamma_1 + i \omega_1) a_1 + 2i V_{k12} A a_2^* \exp(-i \omega_k t) = 0,
$$

$$
\dot{a}_2^* + (\gamma_2 - i \omega_2) a_2^* + 2i V_{k12}^* A^* a_1 \exp(i \omega_k t) = 0.
$$

The solution can be sought in the form

$$
a_1(t) \propto \exp(\Gamma t + i \Omega_1 t), \quad a_2^*(t) \propto \exp(\Gamma t + i \Omega_2 t).
$$

The temporal resonance condition is $\Omega_1 + \Omega_2 = \omega_k$. The amplitudes of the waves will be determined by the mismatches $\Omega_1 - \omega_1$ and $\Omega_2 - \omega_2$, which are the differences between the forced frequencies and the eigenfrequencies, determined by dispersion relation. The sum of the mismatches is $\Delta \omega = \omega_1 + \omega_2 - \omega_k$. We are interested in $\Omega$s that give maximal real $\Gamma$ and expect it when 1 and 2 are symmetrical, so that $\Omega_1 - \omega_1 = \Omega_2 - \omega_2 = \Delta \omega/2$. Consider, for simplicity, $\gamma_1 = \gamma_2$, then

$$
\Gamma = -\gamma \pm \sqrt{4|V_{k12}A|^2 - (\Delta \omega)^2 / 4}.
$$

(3.39)

If the dispersion relation is of the non-decay type, then $\Delta \omega \ll \omega_k \gg |VA|$ and there is no instability. On the contrary, for decay dispersion relations, resonance is possible, $\Delta \omega = \omega_1 + \omega_2 - \omega_k = 0$, so that the growth rate of instability, $\Gamma = 2|V_{k12}A| - \gamma$, is positive when the amplitude is larger than the threshold: $A >$
3.3 Non-linear Schrödinger equation (NSE)

γ/2|V_{12}|, i.e. when the non-linearity overcomes dissipation. The growth rate is maximal for those \(a_1, a_2\) that are in resonance (i.e. \(\Delta \omega = 0\)) and have minimal \((\gamma_1 + \gamma_2)/|V_{12}|\). In the particular case \(k = 0, \omega_0 \neq 0\), decay instability is called parametric instability, since it corresponds to a periodic uniform change in some system parameter. For example, Faraday discovered that a vertical vibration of a container with a fluid leads to the parametric excitation of a standing surface wave \((k_1 = -k_2)\) with half the vibration frequency (the parameter being changed periodically is the gravity acceleration \(g\)). In a simple case of an oscillator it is called parametric resonance, known to any child on a swing who stretches and folds his legs with twice the frequency of the swing (the parameter being changed periodically is the swing effective length \(L\)). In both cases, one varies the frequency \(\sqrt{g/L}\), which is the parameter of the Hamiltonian.

As with any instability, the usual question is what stops the exponential growth and the usual answer is that further non-linearity does that. When the amplitude is not far from the threshold, these non-linear effects can be described in the mean-field approximation as the renormalization of the linear parameters \(\omega_k, \gamma_k\) and of the pumping \(V_{12}A\). The renormalization should be such as to put the wave system back to the threshold, that is to turn the renormalized \(\gamma\) into zero. The frequency renormalization \(\tilde{\omega}_k = \omega_k + \int T_{kk'} |a_{k'}|^2 d\mathbf{k}'\) (see the next section) appears because of the four-wave processes and can take waves out of resonance if the set of wavevectors is discrete, owing to a finite box size (it is a mechanism of instability restriction for finite-dimensional systems like an oscillator – swing frequency decreases with amplitude, for instance). If, however, the box is large enough, then the frequency spectrum is close to continuous and there are waves in resonance for any non-linearity. In this case, the saturation of instability is caused by renormalization of the damping and pumping. The renormalization (increase) of \(\gamma_k\) appears because of the waves of the third generation that take energy from \(a_1, a_2\). The pumping renormalization appears because of the four-wave interaction, for example, (3.38) adds \(-i a_2^* \int T_{1234} a_3 a_4 \delta(\mathbf{k} - \mathbf{k}_3 - \mathbf{k}_4) d\mathbf{k}_3 d\mathbf{k}_4\) to \(\dot{a}_1\).

3.3 Non-linear Schrödinger equation (NSE)

This section is devoted to a non-linear spectrally narrow wave packet. Consideration of the linear propagation of such a packet in Section 3.1.4 taught us the notions of phase and group velocities and caustics. In this section, the account of non-linearity brings equally fundamental notions of the Bogoliubov spectrum of condensate fluctuations, modulational instability, solitons, self-focusing, collapse and wave turbulence.
3 Dispersive waves

3.3.1 Derivation of NSE

Consider a quasi-monochromatic wave packet in an isotropic non-linear medium. Quasi-monochromatic means spectrally narrow, that is the wave amplitudes are non-zero in a narrow region $\Delta k$ of $k$-space around some $k_0$. In this case the processes changing the number of waves (like $1 \rightarrow 2 + 3$ and $1 \rightarrow 2 + 3 + 4$) are non-resonant because the frequencies of all waves are close. Therefore, all the non-linear terms can be eliminated from the interaction Hamiltonian except $H_4$ and the equation of motion has the form

$$\frac{\partial a_k}{\partial t} + i\omega_k a_k = -i \int T_{k_1k_2k_3k_4} a_{k_1} a_{k_2} a_{k_3} b(k + k_1 - k_2 - k_3) \, dk_1 dk_2 dk_3.$$  

(3.40)

Consider now $k = k_0 + q$ with $q \ll k_0$ and expand, similar to (3.18),

$$\omega(k) = \omega_0 + (q v) + \frac{1}{2} q_i q_j \left( \frac{\partial^2 \omega}{\partial k_i \partial k_j} \right) \bigg|_{k_0},$$

where $v = \partial \omega / \partial k$ at $k = k_0$. In an isotropic medium $\omega$ depends only on modulus $k$ and

$$q_i q_j \frac{\partial^2 \omega}{\partial k_i \partial k_j} = q_i q_j \frac{\partial}{\partial k_i} \frac{\partial^2 \omega}{\partial k^2} = q_i q_j \left[ k_i k_j \omega'' - \frac{k_i k_j k}{k^2} \frac{v}{k} \right] = q_i^2 \omega'' + \frac{q_i^2 v}{k}.$$  

Let us introduce the temporal envelope $a_k(t) = \exp(-i\omega_0 t) \psi(q, t)$ into (3.40):

$$\left[ \frac{\partial}{\partial t} - (q v) - \frac{q_i^2 \omega''}{2} - \frac{q_i^2 v}{2k} \right] \psi_q = T \int \psi_{\psi_1} \psi_{\psi_2} \psi_{\psi_3} b(q + q_1 - q_2 - q_3) \, dq_1 dq_2 dq_3.$$  

We assumed the non-linear term to be small, $T \int |a_k|^2 \Delta k^2 \ll \omega_k$, and took it at $k = k_0$. This result is usually represented in $r$-space for $\psi(r) = \int \psi_q \exp(iqr) \, dq$.

The non-linear term is local in $r$-space:

$$\int dr_1 dr_2 dr_3 \psi^*(r_1) \psi(r_2) \psi(r_3) \int dq_1 dq_2 dq_3 \delta(q + q_1 - q_2 - q_3) \times \exp \left[ i(q_1 r_1) - i(q_2 r_2) - i(q_3 r_3) + i(qr) \right] = \int dr_1 dr_2 dr_3 \psi^*(r_1) \psi(r_2) \psi(r_3) \delta(r_1 - r) \delta(r_2 - r) \delta(r_3 - r) = |\psi|^2 \psi.$$  

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and the equation takes the form
\[
\frac{\partial \psi}{\partial t} + v \frac{\partial \psi}{\partial z} = \frac{i \omega''}{2} \frac{\partial^2 \psi}{\partial z^2} - \frac{i v}{2k} \Delta \psi = -i T |\psi|^2 \psi. 
\]
Here the term \( v \frac{\partial \psi}{\partial z} \) is responsible for propagation with the group velocity, \( \omega'' \frac{\partial^2 \psi}{\partial z^2} \) for dispersion and \( \frac{(v / k)}{2} \Delta \psi \) for diffraction. One may ask why in the expansion of \( \omega k + q \) we kept the terms both linear and quadratic in small \( q \). This is because the linear term (which gives \( \frac{\partial \psi}{\partial z} \) in the last equation) can be eliminated by the transition to the moving reference frame \( z \rightarrow z - vt \). We also renormalize the transversal coordinate by the factor \( \sqrt{k_0 \omega'' / v} \) and obtain the celebrated non-linear Schrödinger equation
\[
i \frac{\partial \psi}{\partial t} + \frac{\omega''}{2} \Delta \psi - T |\psi|^2 \psi = 0. \tag{3.41}
\]
Sometimes (particularly for \( T < 0 \)) it is called the Gross–Pitaevsky equation after the scientists who derived it to describe a quantum condensate. This equation is meaningful at different dimensionalities. It may describe the evolution of a three-dimensional packet, as in a Bose–Einstein condensation of cold atoms. When \( r \) is two-dimensional, it may correspond either to the evolution of the packet in a 2D medium (say, for surface waves) or to steady propagation in 3D described by \( iv \psi_z + (v / 2k) \Delta \psi = T |\psi|^2 \psi \), which turns into (3.41) upon relabelling \( z \rightarrow vt \). In a steady case, one neglects \( \psi_z \), since this term is much less than \( \psi_z \). In a non-steady case, this is not necessarily so, since \( \frac{\partial}{\partial t} \) and \( v \frac{\partial}{\partial z} \) might be about to annihilate each other and one is interested in the next terms.

And, finishing with dimensionalities, the one-dimensional NSE corresponds to a stationary two-dimensional case or evolution in a one-dimensional medium.

Different media provide for different signs of the coefficients. Apart from hydrodynamic applications, the NSE also describes non-linear optics. Indeed, Maxwell’s equation for waves takes the form \( \omega^2 - (c^2 / n) \Delta E = 0 \). The refraction index depends on the wave intensity: \( n = 1 + 2 \alpha |E|^2 \). There are different reasons for that dependence (and so different signs of \( \alpha \) may be realized in
3 Dispersive waves

different materials), for example: electrostriction, heating and the Kerr effect (orientation of non-isotropic molecules by the wave field). We consider waves moving mainly in one direction and pass into the reference frame moving with the velocity \( c \), i.e. change \( \omega \rightarrow \omega - ck \). Expanding in small parameters \( k_{\perp}/k \) and \( \alpha|E|^2 \),

\[
ck/\sqrt{n} \approx ck(1 - \alpha|E|^2) + ck_{\perp}^2/2k,
\]

substituting it into

\[
(\omega - ck - ck/\sqrt{n})(\omega - ck + ck/\sqrt{n})E = 0,
\]

and retaining only the first non-vanishing terms in diffraction and non-linearity, we obtain the NSE with \( \nu = c \) after the inverse Fourier transform \( \omega \rightarrow i\partial_t \), \( k_{\perp}^2 \rightarrow -\Delta_{\perp} \). In particular, the one-dimensional NSE describes light in an optical fibre, sound in a beam and pulse in a nerve.

3.3.2 Modulational instability

The simplest effect of the four-wave scattering is frequency renormalization. Indeed, the NSE has a stationary solution as a plane wave with a renormalized frequency \( \psi_0(t) = A_0 \exp(-iTA_0^2t) \) (in quantum physics, this state, coherent across the whole system, corresponds to a Bose–Einstein condensate). Let us describe small perturbations of the condensate. We write the perturbed solution as \( \psi = (A_0 + \tilde{A})e^{-iTA_0^2t + i\varphi} \) and assume the perturbation to be one-dimensional (along the direction which we denote \( \xi \)). The real and imaginary parts of the linearized NSE take the form

\[
\tilde{A}_t + \frac{\omega''}{2}A_0\varphi_{\xi\xi} = 0, \quad \varphi_t = -2TA_0\tilde{A} + \frac{\omega''}{2A_0}\tilde{A}_{\xi\xi}.
\]

If the amplitude of the perturbation is modulated then so is the phase:

\[
\tilde{A} = \propto \exp(ik\xi - t\Omega t), \quad \varphi = \propto \exp(ik\xi - t\Omega t).
\]

The dispersion relation for the perturbations then takes the form:

\[
\Omega^2 = T\omega''A_0^2k^2 + \omega''k^4/4. \quad (3.42)
\]

When \( T\omega'' > 0 \), it is called the Bogoliubov formula for the spectrum of condensate perturbations. We have an instability when \( T\omega'' < 0 \) (the Lighthill criterion).

I first explain this criterion using the language of classical waves and at the end
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\[ \psi = A e^{i \phi} \]

\[ \phi = \omega \xi + K_t \]

\[ \omega = -\phi_t \]

\[ K_t = \phi_{\xi \xi} \]

Figure 3.5 Space dependencies of the wave amplitude, frequency and time derivative of the wavenumber, which demonstrate the mechanism of the modulational instability for \( \omega'' > 0 \) and \( T < 0 \).

of the section I give an alternative explanation in terms of quantum (quasi)-particles. Classically, we define the frequency as minus the time derivative of the phase: \( \phi_t = -\omega \). For a non-linear wave, the frequency is generally dependent on both the amplitude and the wavenumber. The first derivatives of the frequency with respect to the amplitude and the wavenumber are zero at zero. The factors \( T \) and \( \omega'' \) are the second derivatives, respectively, with respect to the amplitude and the wavenumber. That is, instability happens when the surface \( \omega(k, A) \) has a saddle point at \( k = 0 = A \). Intuitively, one can explain the modulational instability in the following way: consider, for instance, \( \omega'' > 0 \) and \( T < 0 \). Introduce the current wavenumber \( K = \phi_{\xi} \), whose time derivative is as follows: \( K_t = \phi_{\xi \xi} = -\omega_{\xi} \). If the amplitude acquires a local minimum as a result of perturbation then the frequency has a maximum there because \( T < 0 \). The local maximum in \( \omega \) means that \( K_t \) changes sign, that is \( K \) will grow to the right of the \( \omega \) maximum and decrease to the left of it. The group velocity \( \omega' \) grows with \( K \) since \( \omega'' > 0 \). Then the group velocity grows to the right and decreases to the left so that the parts separate (as the arrows show) and the perturbation deepens, as shown in Figure 3.5.

The result of this instability can be seen on the beach, where waves coming to the shore are modulated. Indeed, for long water waves \( \omega_k \propto \sqrt{k} \) so that \( \omega'' < 0 \). As opposed to a pendulum and somewhat counterintuitively, the frequency grows with the amplitude and \( T > 0 \); it is related to the change of wave shape from sinusoidal to that forming a sharpened crest, which reaches 120° for sufficiently high amplitudes. A long water wave is thus unstable with respect to longitudinal
Figure 3.6 Disintegration of the periodic wave due to modulational instability as demonstrated experimentally by Benjamin and Feir (1967). The upper photograph shows a regular wave pattern close to a wavemaker. The lower photograph is made some 60 metres (28 wavelengths) away, where the wave amplitude is comparable, but spatial periodicity is lost. The instability was triggered by imposing on the periodic motion of the wavemaker a slight modulation at the unstable sideband frequency; the same disintegration occurs naturally over longer distances. Photograph by J. E. Feir, reproduced from Proc. R. Soc. Lond. A, 299, 59 (1967).

modulations (Benjamin–Feir instability, 1967). The growth rate is maximal for \( k = A_0 \sqrt{-2T/\omega''} \), which depends on the amplitude (Figure 3.6). Still, folklore has it that approximately every ninth wave is the largest. The maximal growth rate is quadratic in the wave amplitude: \( \text{Im} \Omega = T A_0^2 \).

For transverse propagation of perturbations, one has to replace \( \omega'' \) by \( v/k \), which is generally positive so the criterion of instability is \( T < 0 \) or \( \partial \omega/\partial |a|^2 < 0 \), which also means that for instability the wave velocity has to decrease with amplitude. This can be easily visualized: if the wave is transversely modulated then the parts of the front where the amplitude is larger will move slower and
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Figure 3.7 Transverse instability for the velocity decreasing with the amplitude.

further increase the amplitude because of focusing from neighbouring parts, as shown in Figure 3.7.

Let us now find a quantum explanation for the modulational instability. Remember that the NSE (3.41) is a Hamiltonian system \( i\dot{\psi} = \delta\mathcal{H} / \delta\psi^* \) with

\[
\mathcal{H} = \frac{1}{2} \int \left( \omega'' |\nabla \psi|^2 + T |\psi|^4 \right) \, \text{dr}.
\]  

(3.43)

The Lighthill criterion means that the modulational instability happens when the Hamiltonian is not sign-definite. The overall sign of the Hamiltonian is unimportant, as the Hamiltonian dynamics time-reversible and one can always change \( \mathcal{H} \rightarrow -\mathcal{H}, \ t \rightarrow -t \); it is important that different configurations of \( \psi(\mathbf{r}) \) give different signs of the Hamiltonian because its two terms have different signs. As a result, uniform state is unstable with respect to breaking into regions where one of the terms dominate. Consider \( \omega'' > 0 \). Using the quantum language one can interpret the first term in the Hamiltonian as the kinetic energy of (quasi)-particles and the second term as their potential energy. For \( T < 0 \), the interaction is attractive, which leads to the instability. For the condensate, the kinetic energy (or pressure) is balanced by the interaction; a local perturbation with more particles (higher \( |\psi|^2 \)) will make the interaction stronger, which leads to the contraction of perturbation and further growth of \( |\psi|^2 \).

Let us compare the modulational instability due to four-wave interaction with the decay instability due to three-wave interaction. The former presumes Lighthill criterium while the latter takes place for any interaction coefficient \( V_{k_1k_2} \) not turning into zero at the resonant manifold \( \omega(k_1+k_2) = \omega(k_1) + \omega(k_2) \).

The growth rate of the decay instability \( V_{k_1k_2}A \) is linear in the wave amplitude, while the modulational instability appears in the next order, so its growth rate
$TA^2$ is generally smaller. For waves on deep water, decay instability is possible for short (capillary) waves; decays are getting non-resonant starting from some wavelength, so that longer (gravity) waves are subject to modulational instability.

### 3.3.3 Soliton, collapse and turbulence

The outcome of the modulational instability depends on space dimensionality. The breakdown of a homogeneous state may lead all the way to small-scale fragmentation or the creation of singularities. Alternatively, stable finite-size objects may appear as an outcome of instability. As often happens, analysis of conservation laws helps to understand the destination of a complicated process.

Since the NSE (3.41) is Hamiltonian, it conserves the energy (3.43). Since (3.41) describes wave propagation and four-wave scattering, it does not change the number of waves and thus conserves the wave action $N = \int |\psi|^2 \, dr$. The conservation follows from the continuity equation

$$2i\partial_t |\psi|^2 = \omega'' \nabla (\psi^* \nabla \psi - \psi \nabla \psi^*) \equiv -2 \text{div} J.$$  \hfill (3.44)

Note also the conservation of the momentum or total current, $\int J \, dr$, which does not play any role in this section but is important for Exercise 3.7. The symmetries responsible for the three conserved quantities are respectively, invariance of (3.41) with respect to the time shift ($t \rightarrow t + \text{const.}$), space shift ($r \rightarrow r + \text{const.}$) and gauge invariance ($\psi \rightarrow \psi e^{i\alpha}$).

Consider a wave packet characterized by the generally time-dependent size $l$ and the constant value of $N$.

Since one can estimate the typical value of the envelope in the packet as $|\psi|^2 \simeq N/l^d$, then $\mathcal{H} \simeq \omega'' NL^{-2} + T N^2 l^{-d}$ — remember that the second term is negative here. We consider the conservative system, so the total energy is conserved yet we expect the radiation from the wave packet to bring it to the minimum of energy. We wish to understand the direction of the evolution considering it adiabatically slow. Then, in the process of weak radiation, wave action is conserved since it is an adiabatic invariant. This is particularly clear for a quantum system, like a cloud of cold atoms, where $N$ is their number. Whether this minimum corresponds to $l \rightarrow 0$ (which is called self-focusing or collapse)
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The Hamiltonian $H$ as a function of the packet size $l$ under fixed $N$.

Figure 3.8 The Hamiltonian $H$ as a function of the packet size $l$ in three different dimensionalities is shown in Figure 3.8.

(i) $d = 1$. At small $l$ kinetic energy $H \simeq \omega'' N l^{-2}$ dominates and leads to expansion, while attraction $H \simeq -T N^2 l^{-1}$ dominates at large $l$. It is thus clear that a stationary solution must exist with $l \sim \omega'' / T N$, which minimizes the energy. Physically, the pressure of the waves balances the attraction force. Such a stationary solution is called a soliton, short for solitary wave. It is a travelling-wave solution of (3.41), $\psi(x, t) = A(x - ut)e^{i\varphi}$, with the amplitude function just moving and the phase having both a space-dependent travelling part and a uniform non-linear part linearly growing with time: $\varphi(x, t) = f(x - ut) - T A_0^2 t$.

Here, complex $A_0$ and real $u$ are soliton parameters. We substitute the travel solution into (3.41) and separate the real and imaginary parts:

$$A'' = \frac{2T}{\omega''}(A^3 - A_0^2 A) + A f' \left( f' - \frac{2u}{\omega''} \right), \quad \omega'' \left( A' f' + \frac{A f''}{2} \right) = u A'. \quad (3.45)$$

For the simple case of the standing wave ($u = 0$) the second equation gives $f = \text{const.}$, which can be put equal to zero. The first equation can be considered as a Newtonian equation $A'' = -dU/dA$ for the particle with coordinate $A$ in the potential $U(A) = -(T/2\omega'')(A^4 - 2A^2 A_0^2)$ and the space coordinate $x$ replacing the particle’s time. The soliton is a separatrix, that is a solution that requires for particle an infinite time to reach zero, or in original terms where $A \to 0$ as $x \to \pm \infty$. The upper part of Figure 3.9 presumes $T/\omega'' < 0$, that is a case of modulational instability. Let me mention in passing that the separatrix also exists for $T/\omega'' > 0$ but in this case the running wave is a kink, that is a transition between two different values of the stable condensate (the lower part of the figure). The kink is seen as a dip in intensity $|\psi|^2$.

Considering a general case of a travelling soliton (at $T/\omega'' < 0$), one can multiply the second equation by $A$ and then integrate: $\omega'' A^2 f' = u(A^2 - A_0^2)$, where by choosing the constant of integration we defined $A_0$ as $A$ at the point where $f' = 0$. We can now substitute $f'$ into the first equation and get the closed
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Figure 3.9 Energy as a function of the amplitude of a running wave, and the profile of the wave. The upper part corresponds to the case of an unstable condensate, where a steady solution is a soliton, the lower part to a stable condensate, where it is a kink.

The soliton solution has the form:

\[ \psi(x, t) = \sqrt{2A_0} \cosh^{-1} \left( -\frac{2T\omega}{\omega'} \right)^{1/2} A_0(x - ut) e^{i(2x - ut)/2\omega'' - iTA_0^2/\omega'}. \]

Note that the Galilean transformation for the solutions of the NSE appears as \( \psi(x, t) \rightarrow \psi(x - ut, t) \exp(i(2x - ut)/2\omega''). \) In the original variable \( n(r) \), our envelope solitons appear as shown in Figure 3.10.

(ii) \( d = 2, 3 \). When the condensate is stable, there exist stable solitons analogous to kinks, which are localized minima in the condensate intensity. In optics they can be seen as grey and dark filaments in a laser beam propagating through a non-linear medium. The wave (condensate) amplitude turns into zero in a dark filament, which means that it is a vortex, i.e. a singularity of the wave phase, see Exercise 3.6.

When the condensate is unstable, there are no stable stationary solutions for \( d = 2, 3 \). From the dependence \( \mathcal{H}(l) \) shown in Figure 3.8 we expect that the character of evolution will be completely determined by the sign of the Hamiltonian at \( d = 2 \): the wave packets with positive Hamiltonian spread because the wave dispersion (kinetic energy or pressure, in other words) dominates while the wave packets with negative Hamiltonian shrink and collapse. Let me stress that this way of arguing based on the dependence \( \mathcal{H}(l) \) is non-rigorous and
3.3 Non-linear Schrödinger equation (NSE)

3.3.1 Hamiltonian of the envelope

The Hamiltonian of the envelope of the monochromatic wave can be written as

\[ H = \frac{1}{2} \int |\nabla \psi|^2 + \frac{1}{2} \int |\psi|^2 \]

which is valid only for scales much larger than the wavelength of the carrier wave. 

Consider the envelope equation

\[ i \frac{\partial \psi}{\partial t} = \nabla \cdot (\nabla \psi + \nabla \alpha \psi) \]

and its time derivative:

\[ \frac{\partial}{\partial t} \left( i \frac{\partial \psi}{\partial t} \right) = \nabla \cdot (\nabla \nabla \psi + \nabla \nabla \alpha \psi + \nabla \alpha \nabla \psi) \]

This expression suggests at best. A rigorous proof of the fact that the Hamiltonian sign determines whether the wave packet spreads or collapses in 2D is called Talanov’s theorem, which is the expression for the second time derivative of the packet size squared,

\[ l^2(t) = \int |\psi|^2 r^2 \, dr \]

To obtain that expression, differentiate over time using (3.44), then integrate by parts, then differentiate again:

\[ \frac{d^2 l^2}{\omega'' dt^2} = \frac{\partial}{\partial t} \left( i \frac{\partial \psi}{\partial t} \right) = i \int \nabla \cdot (\nabla \psi + \nabla \alpha \psi) \, dr = i \int \nabla \cdot (\nabla \psi + \nabla \alpha \psi) \, dr = \]

\[ - \frac{\omega''}{2} \left( \psi \nabla \Delta \psi^* + \psi^* \nabla \Delta \psi - \nabla \beta (\nabla \beta \psi \nabla \psi^* + \nabla \beta \psi^* \nabla \psi) \right) \]

\[ - T \nabla \alpha |\psi|^4 \, dr = dT \int |\psi|^4 \, dr + 2 \omega'' \int |\nabla |\psi|^2 \, dr = 4H + 2(d-2)T \int |\psi|^4 \, dr. \]

Consider an unstable case with \( T \omega'' < 0 \). We see that indeed for \( d \geq 2 \) one has an inequality \( \partial_t t^2 \leq 4 \omega'' H \) so that

\[ l^2(t) \leq 2 \omega'' H t^2 + C_1 t + C_2 \]

and for \( \omega'' H < 0 \) the packet shrinks to singularity in a finite time (this is the singularity in the framework of NSE, which is itself valid only for the scales much larger than the wavelength of the carrier wave \( 2\pi/k_0 \)). This, in particular,
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describes self-focusing of light in non-linear media. For \( d = 2 \) and \( \omega''H > 0 \), one has dispersive expansion and decay.

**Turbulence with two cascades.** As mentioned, any equation (3.40) that describes only four-wave scattering necessarily conserves both the energy \( H \) and the number of waves (or wave action) \( N \). For waves of small amplitude, the energy is approximately quadratic in wave amplitudes, \( H \approx \int \omega |a_k|^2 \, dk \), as well as \( N = \int |a_k|^2 \, dk \). The existence of two quadratic positive integrals of motion in a closed system means that if such system is subject to external pumping and dissipation, it may develop turbulence consisting of two cascades.

Indeed, imagine that the source at some \( \omega_2 \) pumps \( N_2 \) waves per unit time. It is then clear that for a steady state one needs two dissipation regions in \( \omega \)-space (at some \( \omega_1 \) and \( \omega_3 \)) to absorb the inputs of both \( N \) and \( E \). Conservation laws allow one to determine the numbers of waves, \( N_1 \) and \( N_3 \), absorbed per unit time in the regions of low and high frequencies, respectively. Schematically, solving \( N_1 + N_3 = N_2 \) and \( \omega_1 N_1 + \omega_3 N_3 = \omega_2 N_2 \) we get

\[
N_1 = N_2 \frac{\omega_3 - \omega_2}{\omega_3 - \omega_1}, \quad N_3 = N_2 \frac{\omega_2 - \omega_1}{\omega_3 - \omega_1}.
\]

We see that for a sufficiently large left interval (when \( \omega_1 \ll \omega_2 < \omega_3 \)) most of the energy is absorbed by the right sink: \( \omega_2 N_2 \approx \omega_3 N_3 \). Similarly at \( \omega_1 < \omega_2 \ll \omega_3 \) most of the wave action is absorbed at small \( \omega \): \( N_2 \approx N_1 \). When \( \omega_1 \ll \omega_2 \ll \omega_3 \) we have two cascades with the fluxes of energy \( \epsilon \) and wave action \( Q \). The \( Q \)-cascade towards large scales is called the inverse cascade (Kraichnan, 1967; Zakharov, 1967); it corresponds, somewhat counterintuitively, to a kind of self-organization, i.e. the creation of larger and slower modes out of small-scale fast fluctuations.\(^9\) The limit \( \omega_1 \to 0 \) is well-defined; in this case the role of the left sink can actually be played by a condensate, which absorbs an inverse cascade. Note in passing that consideration of thermal equilibrium in a finite-size system with two integrals of motion leads to the notion of negative temperature.\(^10\)

An important hydrodynamic system with two quadratic integrals of motion is a two-dimensional incompressible ideal fluid. In two dimensions, the velocity \( \mathbf{u} \) is perpendicular to the vorticity \( \omega = \nabla \times \mathbf{u} \), so that the vorticity of any fluid element is conserved by virtue of the Kelvin theorem. This means that the space integral of any function of vorticity is conserved, including \( \int \omega^2 \, d\mathbf{r} \), called...
enstrophy. We can write the densities of the two quadratic integrals of motion, energy and enstrophy, in terms of the velocity spectral density: \( E = \int |v_k|^2 \, dk \) and \( \Omega = \int |k \times v_k|^2 \, dk \). Assume now that we excite turbulence with a force having a wavenumber \( k_2 \) while dissipation regions are at \( k_1, k_3 \). Applying the consideration similar to (3.46), we express the energy dissipation rates \( E_1, E_3 \) via the input rate \( E_2 \):

\[
E_1 = \frac{k_2^2 - k_3^2}{k_2^2 - k_1^2}, \quad E_3 = \frac{k_3^2 - k_1^2}{k_2^2 - k_1^2}.
\]  
(3.47)

We see that for \( k_1 \ll k_2 \ll k_3 \), most of the energy is absorbed by the left sink, \( E_1 \approx E_2 \), and most of the enstrophy is absorbed by the right one, \( \Omega_2 = k_2^2 E_2 \approx \Omega_3 = k_3^2 E_3 \). We conclude that conservation of both energy and enstrophy in two-dimensional flows requires two cascades: that of the enstrophy towards small scales and that of the energy towards large scales (opposite to the direction of the energy cascade in three dimensions). Large-scale motions of the ocean and planetary atmospheres can be considered to be approximately two-dimensional; the creation and persistence of large-scale flow patterns in these systems is probably related to inverse cascades.\(^\text{11}\)

### 3.4 Korteveg–de-Vries (KdV) equation

Here we consider another universal limit: weakly non-linear long waves. In the long-wave limit one may expand frequency in the powers of wavenumber (more accurately, reversibility of a Hamiltonian system means that we expand \( \omega^2 \) in powers of \( k^3 \)). If homogeneous perturbation (\( k = 0 \)) does not cost any energy then this expansion starts from the first term, that is the dispersion relation of such waves is close to acoustic. We derive the respective KdV equation for shallow-water waves. We then consider some remarkable properties of this equation and of such waves.

#### 3.4.1 Waves in shallow water

Linear gravity-capillary waves have \( \omega_k^2 = (gk + \alpha k^3 / \rho) \tanh kh \), see (3.17). That is, for sufficiently long waves (when the wavelength is larger than both \( h \) and \( \sqrt{\alpha / \rho g} \)) their dispersion relation is close to linear:

\[
\omega_k = \sqrt{gh} k - \beta k^3, \quad \beta = \frac{\sqrt{gh}}{2} \left( \frac{h^2}{3} - \frac{\alpha}{\rho g} \right).
\]  
(3.48)
That means that shallow-water waves are similar to acoustic waves with the speed of sound $c = \sqrt{gh}$. In the nineteenth century, J. S. Russel used this formula to estimate the atmosphere height, observing propagation of weather changes, that is of pressure waves. Taking $h \simeq 10$ km, we obtain $c \simeq 320$ m s$^{-1}$. Non-surprisingly, it is comparable with the speed of sound in the air near the surface, $c = \sqrt{\gamma P/\rho}$, with $P \simeq \rho gh$.

When different harmonics propagate with almost the same velocity, one can expect a quasi-simple plane wave propagating in one direction, like that described in Sections ?? and ??`. The main effect will be propagation with the speed $\sqrt{gh}$ without changing form, while small effects of nonlinearity and dispersion will lead to slow changes. Let us derive the equation describing such a wave. From the dispersion relation, we obtain the linear part of the equation: $u_t + \sqrt{gh}u_x = -\beta u_{xxx}$, or in the reference frame moving with the velocity $\sqrt{gh}$ one has $u_t = -\beta u_{xxx}$. To derive the non-linear part of the equation in the long-wave limit, it is enough to consider the first non-vanishing spatial derivative (i.e. the first). The motion is close to one-dimensional so that $u = v_x \gg v_z$. The $z$ component of the Euler equation gives $\partial p/\partial z = -\rho g$ and $p = p_0 + \rho g (\zeta - z)$. Here $\zeta(x,t)$ is again the elevation of the surface, where the pressure is assumed to be $p_0$. Now we substitute the pressure into the $x$ component of the Euler equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} = -g \frac{\partial \zeta}{\partial x}.$$  

In the continuity equation, $h + \zeta$ now plays the role of density:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} (h + \zeta) u = 0.$$  

We differentiate it with respect to time, substitute the Euler equation and neglect the cubic term:

$$\frac{\partial^2 \zeta}{\partial t^2} = \partial_x (h + \zeta) (uu_x + g \zeta_x) + \partial_x u \partial_x (h + \zeta) u$$

$$\approx \frac{\partial^2}{\partial x^2} \left( gh \zeta + \frac{g}{2} \zeta^2 + hu^2 \right). \quad (3.49)$$

Apparently, the right-hand side of this equation contains terms of different orders. The first term describes the linear propagation with velocity $\sqrt{gh}$ while the rest describe a small non-linear effect. Such equations are usually treated by the method called multiple time (or multiple scale) expansion. We actually applied this method in deriving the Burgers and non-linear Schrödinger equations. We assume that $u$ and $\zeta$ depend on two arguments,
3.4 Korteweg–de-Vries (KdV) equation

namely \( u(x - \sqrt{gh} t, t) \), \( \zeta(x - \sqrt{gh} t, t) \), and, that the dependence on the second argument is slow. In what follows, we write the equation for \( u \). Then at the main order \( \partial_t u = -\sqrt{gh} u_x = -g \zeta_x \), that is \( \zeta = u \sqrt{h \over g} \), which is a direct analogue of \( \delta \rho / \rho = u / c \) for acoustics. From now on, \( u_t \) denotes the derivative with respect to the slow time (or simply speaking, in the reference frame moving with velocity \( \sqrt{gh} \)). We obtain from (3.49):

\[
(\partial_t - \sqrt{gh} \partial_x)(\partial_t + \sqrt{gh} \partial_x)u \approx -2 \sqrt{gh} u_{xt} = (3/2) \sqrt{gh} (u^2)_{xx},
\]

that is the non-linear contribution into \( u_t \) is \(-3uu_x/2\). Comparing this with the general acoustic expression (3.57), which is \(- (\gamma + 1) uu_x / 2\), we see that shallow-water waves correspond to \( \gamma \approx 2 \). This is also clear from the fact that the local ‘sound’ velocity is \( \sqrt{g(h + \zeta)} \approx \sqrt{gh} + u/2 = c + (\gamma - 1)u/2 \), see (3.57).

The analogy between shallow-water waves and sound means that there exist shallow-water shocks called bores\(^\text{12}\) and hydraulic jumps. The Froude number \( u^2 / gh \) plays the role of the (squared) Mach number in this case.

Hydraulic jumps can be readily observed in the kitchen sink when water from the tap spreads radially with the speed, exceeding the linear ‘sound’ velocity \( \sqrt{gh} \): the fluid layer thickness suddenly increases, which corresponds to a shock, see Figure 3.11 and Exercise 3.8. This shock is sent back by the sink sides that stop the flow; the jump position is where the shock speed is equal to the flow velocity.\(^\text{13}\) For a weak shock, the shock speed is the speed of ‘sound’ so that the flow is ‘supersonic’ inside and ‘subsonic’ outside. Long surface waves cannot propagate into the interior region, which can thus be called a white hole (as opposed to a black hole in general relativity) with the hydraulic jump playing the role of a horizon. In Figure 3.11 one sees circular capillary ripples propagating inside; these are to be distinguished from the jump itself which is non-circular.

3.4.2 The KdV equation and the soliton

Now we are ready to combine both the linear term from the dispersion relation and the non-linear term just derived. Making a change \( u \rightarrow 2u/3 \) we turn the coefficient at the non-linear term into unity. The equation

\[
u_t + uu_x + \beta u_{xxx} = 0
\]  

(3.50)

has been derived by Korteweg and de Vries in 1895 and is called the KdV equation. Together with the non-linear Schrödinger and Burgers equations, it is a member of an exclusive family of universal non-linear models. It is one-dimensional, like the Burgers equation, and has the same degree of universality.
Namely, the systems with a continuous symmetry (say, translation invariance) spontaneously broken allow for what is called a Goldstone mode with $\omega \to 0$ when $k \to 0$. When wavelength is larger than any other scale in conservative time-reversible centre-symmetrical systems, one expects an acoustic branch of excitations with an analytic function $\omega^2(k^2)$. That function has the expansion in integer powers, $k^2, k^4, \text{etc}$. For long waves moving in one direction one has $\omega_k = ck_1 + C(kl_0)^2$, where $C$ is a dimensionless coefficient of order unity and $l_0$ is some internal scale in the system. For gravity water waves, $l_0$ is the water depth; for capillary waves, it is $\sqrt{\alpha/\rho g}$. We see from (3.48) that, depending on which scale is larger, $\beta$ can be either positive or negative, which corresponds to the waves with finite $k$ moving slower or faster, respectively, than the ‘sound’ velocity $\sqrt{gh}$. Indeed, adding surface tension to the restoring force, one increases the frequency. On the other hand, the finite depth-to-wavelength ratio means that fluid particles move in ellipses (rather than in straight lines) which decreases the frequency. Quadratic non-linearity $\partial_x u^2$ (which occurs in both the Burgers and KdV equations and means effectively the renormalization of the speed of sound) is also pretty general, indeed, it has to be zero for a uniform velocity so that it contains a derivative. An incomplete list of the excitations described by the KdV equation contains acoustic perturbations in a plasma (where $l$ is either the Debye radius of charge screening or the Larmor cyclotron radius in the magnetized plasma), phonons in solids (where $l$ is the distance between atoms) and phonons in helium (in this case, amazingly, the sign of $\beta$ depends on pressure).
3.4 Korteweg–de-Vries (KdV) equation

The KdV equation, has a symmetry $\beta \rightarrow -\beta$, $u \rightarrow -u$ and $x \rightarrow -x$, which makes it enough to consider only positive $\beta$. Let us first look for the travelling waves, that is substitute $u(x - vt)$ into (3.50):

$$\beta u_{xxx} = vu_x - uu_x. \tag{3.51}$$

This equation has a symmetry $u \rightarrow u + w$, $v \rightarrow v + w$ so that integrating it once we can set the integration constant to zero by choosing an appropriate constant $w$ (this is the trivial renormalization of sound velocity due to a uniformly moving fluid). We thus get the equation

$$\beta u_{xx} = -\frac{\partial U}{\partial u}, \quad U(u) = \frac{u^3}{6} - \frac{vu^2}{2} + \text{const.} \tag{3.52}$$

A general solution of this ordinary differential equation can be written in elliptic functions. We don’t need it though to understand the general properties of the solutions and to pick up the special one (i.e. the soliton). Just like in Section 3.3.3, we can treat this equation as a Newtonian equation for a particle in a potential, treating $u$ as a particle coordinate and $x$ as time. We consider positive $v$ since it is a matter of choosing a proper reference frame. Since unrestricted growth would violate the assumption of weak non-linearity, we restrict ourselves by the solution with finite $|u(x)|$. Such finite solutions have $u$ in the interval $(0, 3v)$.

We see that quasi-linear periodic waves exist near the bottom at $u \approx 2v$. Their amplitude is small in the reference frame moving with $-2v$ – in that reference frame they have negative velocity as must be the case for positive $\beta$. Indeed, the sign of $\beta$ is minus the sign of the dispersive correction $-3\beta k^2$ to the group velocity $d\omega / dk$. The soliton, on the contrary, moves with positive velocity in the reference frame where there is no perturbation at infinity (it is precisely the reference frame used in the picture). Solitons are supersonic if the periodic waves are subsonic and vice versa (the physical reason for this is that the soliton should not be able to radiate resonant linear waves).
3 Dispersive waves

As usual, the soliton solution is a separatrix passing through $u = 0$:

$$u(x,t) = 3v \cosh^{-2} \left( \frac{\sqrt{v}}{\sqrt{4\beta}} (x - vt) \right). \quad (3.51)$$

The higher the amplitude, the faster it moves (for $\beta > 0$) and the more narrow it is. Like the argument at the end of Sect. 3.3.2, one can realize that the 1D KdV soliton is unstable with respect to the perpendicular perturbations if its propagation speed $\sqrt{gh} + v$ decreases with the amplitude, that is for $\beta < 0$ when $v < 0$ in (3.51), so that the soliton is subsonic and linear waves are supersonic.

Without friction or dispersion, non-linearity breaks acoustic perturbation. We see that wave dispersion stabilizes the wave like the viscous friction stabilizes the shock front, but the waveform is, of course, different. The ratio of non-linearity to dispersion, $\sigma = uu_x/\beta u_{xxx} \sim ul^2/\beta$, is an intrinsic non-linearity parameter within the KdV equation (in addition to the original ‘external’ non-linearity parameter, the Mach number $u/c$, assumed to be always small). For a soliton, $\sigma \approx 1$, that is non-linearity balances dispersion as was the case with the NSE soliton from Section 3.3.3. This also shows that the soliton is a non-perturbative object; one cannot derive it by starting from a linear travelling waves and treating non-linearity perturbatively. For the Burgers equation, both finite-$M$ and travelling-wave solutions depended smoothly on the respective intrinsic parameter $Re$ and existed for any $Re$. Introduction of $\sigma$ leads to natural questions: Can we assert that any perturbation with $\sigma \ll 1$ corresponds to linear waves? What can one say about the evolution of a perturbation with $\sigma \gg 1$ within the KdV equation?

3.4.3 Inverse scattering transform

It is truly remarkable that the evolution of arbitrary initial perturbation can be studied analytically within the KdV equation. It is done in a somewhat unexpected way by considering a linear stationary Schrödinger equation with the function $-u(x,t)/6\beta$ as a potential, depending on time as a parameter:

$$\left[ -\frac{d^2}{dx^2} - \frac{u(x,t)}{6\beta} \right] \psi = E \psi. \quad (3.52)$$

Positive $u(x)$ could create bound states, that is a discrete spectrum. As has been noticed by Gardner, Green, Kruskal and Miura in 1967, the spectrum $E$ does not depend on time if $u(x,t)$ evolves according to the KdV equation. In other words, the KdV equation describes the iso-spectral transformation of the
quantum potential. To show this, express $u$ via $\Psi$

$$u = -6\beta \left( E + \frac{\Psi_{xx}}{\Psi} \right).$$  

(3.53)

Notice the similarity to the Hopf substitution one uses for the Burgers equation, $v = -2\nu\phi_t/\phi$. There is one more derivative in (3.53) because there is one more derivative in the KdV equation – despite the seemingly naive and heuristic nature of such thinking, it is precisely the method by which Miura came to suggest (3.53). Now, substitute (3.53) into the KdV equation and derive

$$\psi^2 \frac{dE}{dt} = 6\beta \partial_x \left[ (\psi \partial_x - \Psi_x)(\Psi_t + \Psi_{xxx} - \Psi_x(u + E)/2) \right].$$  

(3.54)

Integrating it over $x$ we get $dE/dt = 0$ since $\int_{-\infty}^{\infty} \psi^2 \, dx$ is finite for a bound state. The eigenfunctions evolve according to the equation that can be obtained by twice integrating (3.54) and setting the integration constant to zero because of the normalization:

$$\Psi_t + \Psi_{xxx} - \Psi_x (u + E)/2 = 0.$$  

(3.55)

From the viewpoint of (3.52), the soliton is the well with exactly one level, $E = v/8\beta$, which could be checked directly. For distant solitons, one can define energy levels independently. For different solitons, velocities are different and they will generally have collisions. Since the spectrum is conserved, after all the collisions we have to have the same solitons. Since the velocity is proportional to the amplitude, the final state of the perturbation with $\sigma \gg 1$ must look like a linearly ordered sequence of solitons: the quasi-linear waves that correspond to a continuous spectrum are left behind and eventually spread. One can prove this and analyze the evolution of an arbitrary initial perturbation using the inverse
scattering transform (IST) method:

\[
\begin{align*}
  u(x, 0) \rightarrow \Psi(x, 0) &= \sum a_n \Psi_n(x, 0) + \int a_k \Psi_k(x, 0) \, dk \\
  \rightarrow \Psi(x, t) &= \sum a_n \Psi_n(x, t) + \int a_k \Psi_k(x, 0) e^{-i\omega_k t} \, dk \\
  \rightarrow u(x, t).
\end{align*}
\] (3.56)

The first step is to find the eigenfunctions and eigenvalues in the potential \( u(x, 0) \). The second (trivial) step is to evolve the discrete eigenfunctions according to (3.55) and the continuous eigenfunctions according to frequency. The third (non-trivial) step is to solve an inverse scattering problem, that is to restore the potential \( u(x, t) \) from its known spectrum and the set of (new) eigenfunctions.\(^{15}\)

Considering weakly non-linear initial data (\( \sigma \ll 1 \)), one can treat the potential energy as a perturbation in (3.52) and use the results from the quantum mechanics of shallow wells. Remember that the bound state exists in 1D if the integral of the potential is negative, that is in our case when the momentum \( \int u(x) \, dx > 0 \) is positive, i.e., the perturbation is supersonic. The subsonic small perturbation (with a negative momentum) does not produce solitons, it produces subsonic quasi-linear waves. On the other hand, however small the non-linearity \( \sigma \) of the initial perturbation with positive momentum is, the soliton (an object with \( \sigma \sim 1 \)) will necessarily appear in the course of evolution. The amplitude of the soliton is proportional to the energy of the bound state, which is known to be proportional to \( (\int u \, dx)^2 \) in the shallow well.

The same IST method was applied to the 1D non-linear Schrödinger equation by Zakharov and Shabat in 1971. Now, the eigenvalues \( E \) of the system

\[
\begin{align*}
  i \partial_t \psi_1 + \psi \psi_2 &= E \psi_1 \\
  -i \partial_t \psi_2 - \psi^* \psi_1 &= E \psi_2
\end{align*}
\]

are conserved when \( \psi \) evolves according to the 1D NSE. Like the KdV equation, one can show that within the NSE, an arbitrary localized perturbation evolves into a set of solitons and a diffusing quasi-linear wave packet.

The reason why universal dynamic equations in one space dimension happen to be integrable may be related to their universality. Indeed, in many different classes of systems, weakly non-linear long-wave perturbations are described by the Burgers or KdV equations and quasi-monochromatic perturbations by the NSE. Those classes may happen to contain degenerate integrable cases; then integrability exists for the limiting equations as well. These systems are actually
Exercises

3.1 Why is it that a beachcomber (a long water wave rolling upon the beach) usually comes to the coast being almost parallel to the coastline even when the wind blows at an angle?

3.2 A quasi-monochromatic packet of waves contains $N$ crests and wells propagating along the fluid surface. How many ‘up and down’ motions does a light float undergo while the packet passes? Consider two cases: (i) gravity waves on deep water, (ii) capillary waves on deep water.

3.3 Dropping a stone into the deep water, one could see, after a little while, waves propagating outside an expanding circle of fluid at rest. Sketch a snapshot of the wave crests. What is the velocity of the boundary of the quiescent fluid circle?

3.4 The existence of the stable small-amplitude waves that are described by (3.32) cannot be taken for granted. Consider a general form of the quadratic Hamiltonian

$$H_2 = \int \left[ A(k) |b_k|^2 + B(k) (b_kb_{-k} + b_k^*b_{-k}^*) \right] dk.$$  \hspace{1cm} (3.57)

(i) Find the linear transformation (the Bogoliubov u–v transformation) $b_k = u_k a_k + v_k a_k^*$ that turns (3.57) into (3.32).

(ii) Consider the case when the even part $A(k) + A(-k)$ changes sign on some surface (or line) in $k$-space while $B(k) \neq 0$ there. What does this mean physically? In this case, what is the simplest form that the quadratic Hamiltonian $H_2$ can be turned into?

3.5 Show that the power dispersion relation $\omega \propto k^\alpha$ is of the decay type if $\alpha \geq 1$, i.e. it is possible to find such $\mathbf{k}_1, \mathbf{k}_2$ that $\omega(\mathbf{k}_1 + \mathbf{k}_2) = \omega_1 + \omega_2$. Consider two-dimensional space $\mathbf{k} = \{k_x, k_y\}$. Hint: $\omega_k$ is a concave surface and the resonance condition can be thought of as an intersection of some two surfaces.

3.6 Consider the case of a stable condensate in 3D and describe a solution of the NSE equation (3.41) having the form

$$\psi = A e^{-i\Lambda^2 t + \varphi} f(r/r_0),$$
where $r$ is the distance from the vortex axis and $\varphi$ is a polar angle. Are the parameters $A$ and $r_0$ independent? Describe the asymptotics of $f$ for small and large distances. Why is it called a vortex?

3.7 Consider a discrete spectral representation of the Hamiltonian of the 1D NSE in a finite medium:

$$
H = \sum_{m} \beta |m|^2 |a_m|^2 + \left( T/2 \right) \sum_{ikm} a_i a_k^* a_{i+k-m}^* a_m.
$$

Take only three modes $m = 0, 1, -1$. Describe the dynamics of such a three-mode system.

3.8 Calculate the energy dissipation rate per unit length of the hydraulic jump. Fluid flows with the velocity $u_1$ in a thin layer of height $h_1$ such that the Froude number slightly exceeds unity: $u_1^2/gh_1 = 1 + \epsilon$, $\epsilon \ll 1$.

3.9 If one ought to take into account both dissipation and dispersion of a sound wave, then the so-called KdV–Burgers equation arises:

$$
u_t + uu_x + \beta u_{xxx} - \mu u_{xx} = 0. \tag{3.58}
$$

For example, it allows one to describe the influence of dispersion on the structure of a weak shock wave. Consider a travelling solution $u_0(x - vt)$ of this equation, assuming zero conditions at $+\infty$: $u_0 = u_{0x} = u_{0xx} = 0$. Sketch the form of $u_0(x)$ for $\mu \ll \sqrt{\beta v}$ and for $\mu \gg \sqrt{\beta v}$.

3.10 Find the dispersion relation of the waves on the boundary between two fluids, one flowing and one still, in the presence of gravity $g$ and the surface tension $\alpha$. Describe possible instabilities. Consider, in particular, the cases $\rho_1 > \rho_2$, $v = 0$ (inverted gravity) and $\rho_1 \ll \rho_2$ (wind upon water).

3.11 We have seen in Sections ??,3.1.1 that propagating wave can induce net fluid flow called Stokes drift. Apparently, standing wave cannot induce a drift. But what if we have a velocity field as a linear superposition of two standing waves: $v(r, t) = \sin(\cot t) V(r) + \sin(\cot + \varphi) U(r)$? Velocity in every point still averages to zero. Integrating the trajectory of a fluid particle over
the period and find out if the net drift is possible. Assume for simplicity that the velocity gradients are much smaller than $\omega$. 