Transport from the Fluid/Gravity Correspondence

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What is hydrodynamics?

Late time effective theory of interacting theories near thermal equilibrium.

A more general definition abandons the concept of late time. Instead we just restrict the effective theory to dynamics of conserved currents only. (This assumes that the algebra of conserved currents is closed on the subspace of near-thermal states)

For a conserved $U(1)$ current:

$$\partial_\mu J^\mu = 0$$

Continuity equation (charge conservation):

$$\partial_t J^0 = \vec{\nabla} \cdot \vec{J}$$

can be viewed as initial value problem for time evolution of the charge density $J^0$ with some initial condition $J^0(t = t_0)$.

Yet, the continuity equation cannot be solved as such. We need extra inputs and that are three functions $\vec{J}(t)$.

Similarly for the energy-momentum conservation:

$$\partial_\mu T^{\mu\nu} = 0, \quad \text{or} \quad \partial_t T^{0\nu} = \nabla_i T^{i\nu}$$

$T^{00} = \epsilon$ – energy density of the fluid and $T^{0i} \sim u^i$ is the velocity of the fluid, while $T^{ij}$ are missing data. Equation of state relates energy density to pressure $P = T^{ii}$, thus reducing the number of required inputs.
Constitutive Relations for $U(1)$ current

Data have to be provided on $\vec{J}^i (T^{ij})$ in order to close the dynamical equations.

So, $\vec{J}$ has to be related to $J^0$ (and $T^{ij}$ to $T^{0j}$)

Linear relation (in the absence of external fields): $\vec{J} \sim \vec{\nabla} J^0$ ($\vec{\nabla}$ is the only vector)

Most general linear constitutive relation

$$\vec{J}(t, x) = \int_{t', x'} D(t - t', x - x') \vec{\nabla} J^0(t', x')$$

$$\vec{J}(\omega, q) = D(\omega, q^2) \vec{q} J^0(\omega, q)$$

$D(t)$ is called "Memory Function". $D(\omega)$ is called "Transport Coefficient Function (TCF)". It contain a wealth (infinite amount) of info about transport coefficients to be determined from the microscopic theory. It generalises the concept of diffusion constant.

Causality: $D(t - t') \sim \theta(t - t')$

Causality is related to short time scales, which is probed by hydro evolution at early times. Early times are important. There have been a lot of talk about early "hydronisation" without isotropisation. Most of the entropy is also produced at early times. Causality constraint is a necessary UV completion of the usual late time hydrodynamics.
Define $\Pi^{ij}$ as a traceless part of $T^{ij}$ (also assume Landau frame)

$$\Pi^{ij}(t, x) = \int_{t', x'} \eta(t-t', x-x') \nabla^{i}T^{0j}(t', x') + \int_{t', x'} \zeta(t-t', x-x') \nabla^{i} \nabla^{j} \nabla^{k}T^{0k}(t', x')$$

$$\Pi^{ij} = \eta(\omega, q^2) q_{i} u_{j}(\omega, q) + \zeta(\omega, q^2) q_{i} q_{j} q_{k} u_{k}(\omega, q)$$

**Bulk viscosity is zero**

$\zeta$ is a second shear viscosity

- introduced by E. Shuryak, M.L., D80 (2009) 065026 $\zeta = 0$
- Yanyan Bu and M.L., arXiv:1406.7222 (PRD), arXiv:1409.3095 (JHEP), $\zeta \neq 0$
Non-linear constitutive relation:

\[ \vec{J}(t, x) = D \vec{\nabla} \vec{J}^0 + \#(\vec{\nabla} \vec{J}^0)^2 + \text{infinitely many terms} \]

\[ \vec{J}(t) = \int_{t_0}^{t} D(t - t') \vec{\nabla} \vec{J}^0(t') \, dt' \]

What about the low limit? We pretend to know \( \vec{J}^0 \) at \( t \geq t_0 \). So, we have no choice but to limit the integral

\[ \vec{J}(t) = \vec{J}(t_0) + \int_{t_0}^{t} D(t - t') \vec{\nabla} \vec{J}^0(t') \, dt' \]

Here \( \vec{J}(t_0) \) is an initial condition for the current.

When \( D \) is not known from a microscopic theory or experiment, it has to be modelled.
Hydro Model A: Diffusion constant/Navier-Stokes

**Instantaneous response**

\[ D(t - t', x - x') = D_0(x - x') \delta(t - t') \]

Then

\[ \vec{J}(t, x) = \int_{x'} D_0(x - x') \vec{\nabla}J^0(t, x') \]

If we further assume that \( D_0 \) is local \( D_0(x - x') = D_0 \delta(x - x') \),

\[ \vec{J}(t, x) = D_0 \vec{\nabla}J^0(t, x) \]

which is the usual diffusive approximation. Notice that \( \vec{J}(t = t_0) \neq 0 \)

For the spatial components of the energy-momentum tensor:

\[ \Pi^{ij}(t, x) = \eta_0 \nabla^i u^j(t, x) \quad i \neq j \quad \text{Navier Stokes} \]
Hydro Model B: Relaxation time approximation

\[ D(t - t', x - x') = \frac{D_0}{\tau} e^{-(t-t')/\tau} \delta(x - x'), \quad D(\omega) \sim \frac{1}{\omega - i/\tau} \]

\[ \partial_t \tilde{J}(t, x) = \frac{1}{\tau} \left[ D_0 \nabla \tilde{J}^0(t, x) - \tilde{J}(t, x) + \tilde{J}(t_0, x) \right] \]

\( \tilde{J}(t_0, x) \) is usually assumed to be zero.

A similar construction for \( T^{ij}(t, x), \quad \partial_t \Pi^{ij}(t, x) = \cdots \) is frequently referred to as Israel-Stewart hydrodynamics. It reduces to NS in the limit \( \tau \to 0 \).
Is there a "better" model?


proposed 2-pole approximation (Model C):

\[ D(t - t') = d_1 e^{-(t-t')\omega_1} + d_2 e^{-(t-t')\omega_2} \]

What is the right memory function \( D \)? How to compute it from a microscopic theory?
For a holographically defined microscopic theory (not QCD), \( D \) can be computed using the fluid-gravity correspondence.
The correct structure, more or less, is

\[ D(t) = \theta(t) \sum_{n=0}^{\infty} d_n e^{-t\omega_n} \]


What about the initial conditions $J^0(t_0)$ and $\vec{J}(t_0)$? So far, we pretended that $J^0(t_0)$ is provided as an experimental input (fitted to experiment). At the same time $\vec{J}(t_0)$ is usually simply modelled.

Is it justified to treat them so differently and independently?

One option is to assume that the system emerges from far out of equilibrium and then "hydronises" with the initial conditions $J^0(t_0)$ and $\vec{J}(t_0)$ being random and totally independent.

From a holographic perspective one would have to consider black-hole formation, which is a process dual to hydronization, as a result of a collision/collapse.

many people/many papers
An alternative possibility is that the system "remembers" its \( t < t_0 \) history.

A way to study this is to start at \( t = -\infty \) with a system at equilibrium and turn on an external field which would take the system not too far out of equilibrium, that is the system all the time remains in the hydro regime

For e/m current we can turn on an external electric field \( \vec{E} \).

In presence of external electric field, the constitutive relation generalises to

\[
\tilde{J}(t) = \int_{-\infty}^{t} D(t - t') \vec{\nabla} J^0(t') \, dt' + \int_{-\infty}^{t} \tilde{\sigma}_e(t - t') \vec{E}(t') \, dt'
\]

with \( J^0(t = -\infty) = 0 \). The last term generalises the Ohm's law.

Switch off the external field at \( t = 0 \) and let the system relax towards equilibrium

\[
\tilde{J}(t > 0) = \int_{0}^{t} D(t - t') \vec{\nabla} J^0(t') \, dt' + \tilde{J}_H(t)
\]

\[
\tilde{J}_H(t) = \int_{-\infty}^{0} D(t - t') \vec{\nabla} J^0(t') \, dt' + \int_{-\infty}^{0} \tilde{\sigma}_e(t - t') \vec{E}(t') \, dt'
\]

The "history" current depends on time for \( t > 0 \) and it is not clear if \( \tilde{J}_H \) can be rendered into \( t \)-independent. Thus this hydro presumably cannot be solved as initial value problem.
Gradient expansion of the TCFs

\[ \tilde{J}(t) = \int_{t'} D(t-t') \tilde{\nabla} J^0(t') + \int_{t'} \tilde{\sigma}_e(t-t') \tilde{E}(t') \]

\[ D(t-t') = \int \frac{d\omega}{2\pi} D(\omega) e^{-i\omega(t-t')} = \int \frac{d\omega}{2\pi} D(i\partial_t) e^{-i\omega(t-t')} = D(i\partial_t) \delta(t-t') \]

\[ \tilde{J}(t, x) = D(i\partial_t, \nabla^2) \tilde{\nabla} J^0(t, x) + \sigma_e(i\partial_t, \nabla^2) \tilde{E}(t, x) \]

**Gradient (small momenta) expansion:**

\[ D(i\partial_t, \nabla^2) = D_0 [1 + i\tau \partial_t + \lambda \nabla^2 + ...], \quad \sigma_e(i\partial_t, \nabla^2) = \sigma_0 [1 + i\tau_\sigma \partial_t + ...] \]

\[ \sigma_0 \text{ is a DC conductivity.} \]

**Another way of steering the system out of equilibrium is by magnetic field**

\[ \tilde{J}(t, x) = D(i\partial_t, \nabla^2) \tilde{\nabla} J^0(t, x) + \sigma_e(i\partial_t, \nabla^2) \tilde{E}(t, x) + \sigma_m(i\partial_t, \nabla^2) \tilde{\nabla} \times \tilde{B}(t, x) \]

**This is the most general linear constitutive relation.**
Neutral conformal fluids in a weakly curved 4d background

A way to steer a neutral fluid out of equilibrium is to shake it by an external metric perturbation.

Most general constitutive relation with weakly curved metric \( (h_{\mu \nu} \sim u_i) \)

E. Shuryak, M.L., D80 (2009) 065026

\[
\Pi_{\mu \nu} = -\eta \nabla_{\mu} u_{\nu} - \zeta \nabla_{\mu} \nabla_{\nu} u + \kappa u^{\alpha} u^{\beta} C_{\mu \alpha \nu \beta} + \rho u^{\alpha} \nabla^{\beta} C_{\mu \alpha \nu \beta} + \xi \nabla^{\alpha} \nabla^{\beta} C_{\mu \alpha \nu \beta} - \theta u^{\alpha} \nabla_{\alpha} R_{\mu \nu},
\]

\( C_{\mu \alpha \nu \beta}, R_{\mu \nu} \) are the Weyl and Ricci tensors of \( h_{\mu \nu} \).

\( \kappa(\omega, q^2), \rho(\omega, q^2), \xi(\omega, q^2), \theta(\omega, q^2) \) - Gravitational Susceptibilities of the Fluid (GSFs).

All GSFs contribute to two-point correlators.
\[
\bar{J}(\omega, \vec{q}) = -\mathcal{D} \left( \omega, q^2 \right) \vec{iq} \rho(\omega, \vec{q}) + \sigma_e \left( \omega, q^2 \right) \vec{E}(\omega, \vec{q}) + \sigma_m \left( \omega, q^2 \right) \vec{iq} \times \vec{B}(\omega, \vec{q}).
\]

\[
\mathcal{D} = D_0 \left[ 1 + i \tau \partial_t + \lambda \nabla^2 + \ldots \right] = \frac{1}{2} + \frac{1}{8} \pi i \omega + \frac{1}{48} \left[ -\pi^2 \omega^2 + q^2 \left( 6 \log 2 - 3\pi \right) \right] + \ldots,
\]

\[
\sigma_e = \sigma_e^0 \left[ 1 + i \sigma_\tau \partial_t + \ldots \right] = 1 + \frac{\log 2}{2} i \omega + \frac{1}{24} \left[ \pi^2 \omega^2 - q^2 \left( 3\pi + 6 \log 2 \right) \right] + \ldots,
\]

\[
\sigma_m = 0 + \frac{1}{16} i \omega \left( 2\pi - \pi^2 + 4 \log 2 \right) + \ldots.
\]

**Blue terms are new!**  

\[
\sigma_m^0 > 0 \text{ in a pure QED plasma with one Dirac fermion at one loop level}
\]


\[
\sigma_m^0 = 0 \text{ based on Boltzmann equations}
\]

J. Hong and D. Teaney, (2010)
QCD $\rightarrow \mathcal{N} = 4$ SYM (CFT). Strong coupling (and large $N_c$) $\rightarrow$ AdS/CFT $\rightarrow$ SUGRA on AdS$_5$. CFT at finite Temperature $\leftrightarrow$ AdS Black Hole

Bulk fields (gravitons, photons, etc) propagate signals from the horizon to the boundary, where the hologram is captured.

The bulk acts as a highly nonlinear dispersive medium.

There is no dissipation in the bulk.

All dissipative effects take place at the horizon.
Neutral Flow from Fluid-Gravity correspondence

5d GR with negative cosmological constant:

\[ S = \frac{1}{16\pi G_N} \int d^5x \sqrt{-g} \left( R + 12 \right), \]

Einstein Equations

\[ E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} R - 6g_{\mu\nu} = 0. \]

Solution: Boosted Black Brane in asymptotic AdS$_5$

\[ ds^2 = -2u_\mu dx^\mu dr - r^2 f(br) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu, \]

\[ f(r) = 1 - 1/r^4 \quad \text{and} \quad P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu \]

Hawking temperature

\[ T = \frac{1}{\pi b}, \]

Promote $u_i$ and $b$ into a slowly varying functions of boundary coordinates $x^\alpha$

$$ds^2 = -2u_\mu(x^\alpha)dx^\mu dr - r^2 f(b(x^\alpha)r) u_\mu(x^\alpha) u_\nu(x^\alpha) dx^\mu dx^\nu + r^2 \mathcal{P}_{\mu\nu}(x^\alpha) dx^\mu dx^\nu,$$

Use gradient expansion of the fields $u(x) = u_0 + \delta x \nabla u$ and $b(x) = b + \delta x \nabla b$ to set up a perturbative procedure

The stress tensor for the dual fluid

$$T^\mu_\nu = \lim_{r \to \infty} \tilde{T}^\mu_\nu(r) ; \quad \tilde{T}^\mu_\nu(r) \equiv r^4 \left( K^\mu_\nu - K\gamma^\mu_\nu + 3\gamma^\mu_\nu - \frac{1}{2} G^\mu_\nu \right),$$

$$T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + \Pi^{\mu\nu}_{\text{NS}} + \tau_R (u \nabla) \Pi^{\mu\nu}_{\text{NS}} + O[(\nabla u)^2]$$


$$\frac{\eta_0}{s} = \frac{1}{4 \pi}, \quad \tau_R = 2 - \log(2)$$

R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, M. A. Stephanov

JHEP 0804, 100 (2008)

We do it somewhat differently, linearizing in the velocity amplitude

\[ u_\mu(x^\alpha) = (-1, \epsilon \beta_i(x^\alpha)) + \mathcal{O}(\epsilon^2), \quad b(x^\alpha) = b_0 + \epsilon b_1(x^\alpha) + \mathcal{O}(\epsilon^2), \]

"seed" metric, i.e., a linearized version of the BH metric

\[
\begin{align*}
ds^2_{\text{seed}} &= 2drdv - r^2f(r)dv^2 + r^2dx^2 - \epsilon \left[ 2\beta_i(x^\alpha)drdx^i + \frac{2}{r^2}\beta_i(x^\alpha)dvdx^i + \frac{4}{r^2}b_1(x^\alpha)dv^2 \right] + \mathcal{O}(\epsilon^2), \\
\end{align*}
\]

\[
\begin{align*}
ds^2 &= ds^2_{\text{seed}} + ds^2_{\text{corr}[\beta]} \\
&\quad \text{gauge fix } \ g_{rr} = 0, \quad g_{r\mu} \propto u_\mu
\end{align*}
\]

\[
\begin{align*}
ds^2_{\text{corr}} &= \epsilon \left( -3h drdv + \frac{k}{r^2} dv^2 + r^2h d\vec{x}^2 + \frac{2}{r^2}j_i dvdx^i + r^2\alpha_{ij} dx^i dx^j \right)
\end{align*}
\]

\(h[\beta], \ k[\beta], \ j[\beta], \ \alpha[\beta]\) are to be found by solving the Einstein equations.

Boundary cond: no singularities

\[
\begin{align*}
h < \mathcal{O}(r^0), \quad k < \mathcal{O}(r^4), \quad j_i < \mathcal{O}(r^4), \quad \alpha_{ij} < \mathcal{O}(r^0).
\end{align*}
\]
Einstein equations for the metric corrections

**Dynamical equations:**

\[ E_{rr} = 0 : \quad 5 \partial_r h + r \partial_r^2 h = 0. \]
\[ E_{rv} = 0 : \quad 3 r^2 \partial_r k = 6 r^4 \partial \beta + r^3 \partial_v \partial \beta - 2 \partial j - r \partial_r \partial_j - r^3 \partial_i \partial_j \alpha_{ij} \]
\[ E_{ri} = 0 : \quad -\partial_r^2 j_i = \left( \partial^2 \beta_i - \partial_i \partial \beta \right) + 3r \partial_v \beta_i - \frac{3}{r} \partial_r j_i + r^2 \partial_r \partial_j \alpha_{ij}. \]
\[ E_{ij} = 0 : \quad (r^7 - r^3) \partial_r^2 \alpha_{ij} + (5r^6 - r^2) \partial_r \alpha_{ij} + 2r^5 \partial_v \partial_r \alpha_{ij} + 3r^4 \partial_v \alpha_{ij} \]
\[ + r^3 \left\{ \partial^2 \alpha_{ij} - \left( \partial_i \partial_k \alpha_{jk} + \partial_j \partial_k \alpha_{ik} - \frac{2}{3} \delta_{ij} \partial_k \partial_l \alpha_{kl} \right) \right\} \]
\[ + \left( \partial_i j_j + \partial_j j_i - \frac{2}{3} \delta_{ij} \partial j \right) - r \partial_r \left( \partial_i j_j + \partial_j j_i - \frac{2}{3} \delta_{ij} \partial j \right) \]
\[ + 3r^4 \left( \partial_i \beta_j + \partial_j \beta_i - \frac{2}{3} \delta_{ij} \partial \beta \right) + r^3 \partial_v \left( \partial_i \beta_j + \partial_j \beta_i - \frac{2}{3} \delta_{ij} \partial \beta \right) = 0. \]

**Constraint equations**

\[ E_{vv} = 0 \text{ and } E_{vi} = 0 \quad \rightarrow \quad \partial_\mu T^{\mu \nu} = 0 \]
Maxwell field in Schwarzschild-$AdS_5$ geometry (probe approximation)

\[
S = - \int d^5x \sqrt{-g} \frac{1}{4} e^2 (F^V)_{MN}(F^V)^{MN} + S_{c.t.}
\]

Maxwell equations

\[
\text{EQ}^N := \nabla_M F^{MN} = 0
\]

4 dynamical eqns $\text{EQ}^\mu = 0 \rightarrow$ transport, $\text{EQ}^r = 0 \rightarrow$ current conservations.

Near the conformal boundary $r = \infty$ the solution is expandable in a series ($A_r = 0$)

\[
A^\mu_r(r, x_\alpha) = A^{(0)}(x_\alpha) + \frac{A^{(1)}(x_\alpha)}{r} + \frac{A^{(2)}(x_\alpha)}{r^2} + \frac{B^{(2)}(x_\alpha)}{r^2} \log r^{-2} + \mathcal{O}\left(\frac{\log r^{-2}}{r^3}\right),
\]

The boundary current (using the holographic dictionary)

\[
J^\mu = -\eta^{\mu\nu} \left(2A^{(2)}_\nu + 2B^{(2)}_\nu + \eta^{\sigma t} \partial_\sigma F^{(0)}_{tv} \right).
\]
$\text{EQ}^{\mu} = 0$ admit the most general static homogeneous solutions

$$A_\mu = A_\mu^{(0)} + \frac{\rho}{2r^2}\delta_{\mu t}, \quad A_\mu^{(0)} = \text{const}, \; \rho = \text{const}$$

The boundary theory is a static uniformly charged plasma with no external fields

$$J^0 = \rho, \quad \vec{J} = 0$$

Next, following the spirit of S. Bhattacharyya, V. E. Hubeny, S. Minwalla, and M. Rangamani, (2008)

$$A_\mu^{(0)} \to A_\mu^{(0)}(x_\alpha), \quad \rho \to \rho(x_\alpha).$$

The solution has to be amended:

$$A_\mu(r, x_\alpha) = A_\mu^{(0)}(x_\alpha) + \frac{\rho(x_\alpha)}{2r^2}\delta_{\mu t} + a_\mu(r, x_\alpha)$$

Solve for $a$ (bulk-to-boundary propagator). $a[A^0, \rho]$ is linear both in $A^0$ and $\rho$

Different from approaches based on two-point correlators, which impose current conservation (on-shellness)
$U_V(1) \times U_A(1)$: Anomaly-induced transport

Axial current is not conserved

$$\partial_\mu J^\mu = 0, \quad \partial_\mu J_5^\mu = 12k \vec{E} \cdot \vec{B}$$

General form of constitutive relations

$$\vec{J} = \vec{J}[\rho, \rho_5, T, \vec{E}, \vec{B}]; \quad \vec{J}_5 = \vec{J}_5[\rho, \rho_5, T, \vec{E}, \vec{B}]$$

The one we considered so far

$$\vec{J} = \gamma_1 \vec{\nabla} \rho + \gamma_2 \vec{\nabla} \rho_5 + \gamma_3 \vec{E} + \gamma_4 (\rho \vec{B}) + \gamma_5 \vec{\nabla} \times \vec{B} + \gamma_6 (\vec{\nabla} \rho \times \vec{\nabla} \rho_5) + \gamma_7 (\vec{E} \times \vec{\nabla} \rho)$$
$$+ \gamma_8 (\vec{B} \times \vec{\nabla} \rho) + \gamma_9 (\vec{E} \times \vec{\nabla} \rho_5) + \gamma_{10} (\vec{B} \times \vec{\nabla} \rho_5) + \gamma_{11} (\rho \vec{E} \times \vec{B}) + \gamma_{12} \vec{\nabla} \left( \vec{B} \cdot \vec{\nabla} \rho_5 \right)$$
$$+ \gamma_{13} \vec{\nabla} \left( \vec{B} \cdot \vec{\nabla} \rho \right) + \gamma_{14} \vec{\nabla} (\rho \vec{E}^2) + \gamma_{15} (\rho \vec{B}^2) + \gamma_{16} (\rho \vec{B}) + \gamma_{17} \vec{\nabla} (\vec{E} \cdot \vec{\nabla} \rho) + \gamma_{18} (\rho \vec{E})$$
$$+ \gamma_{19} \vec{\nabla} (\vec{E} \cdot \vec{\nabla} \rho_5) + \gamma_{20} (\rho_5 \vec{E}),$$

$$\gamma_i = \gamma_i \left( \partial_t, \vec{\nabla}, \vec{E} \cdot \vec{\nabla}, \vec{B} \cdot \vec{\nabla}; \vec{E}^2, \vec{B}^2, \vec{E} \cdot \vec{B} \right)$$
• $\mathbf{J} \propto \mathbf{B}$  
  CME
• $\mathbf{J} \propto \partial_t \mathbf{B}$  
  time relaxation in CME
• $\mathbf{J} \propto \mathbf{B}^2 \mathbf{B}$ & $(\mathbf{B} \mathbf{E}) \mathbf{B}$  
  the first nonlinear corrections to CME
• $\mathbf{J} \propto (\mathbf{B} \times \mathbf{E}) \times \mathbf{E}$  
  $E^2$ correction to CME and chiral electric effect (CEE)
• $\mathbf{J} \propto \mathbf{E} \times \mathbf{B}$  
  chiral Hall current
• $\mathbf{J} \propto \mathbf{B} \times (\mu \nabla \mu_5 + \mu_5 \nabla \mu)$  
  Hall diffusion
• $\mathbf{J} \propto \mathbf{E} \times \nabla \mu$  
  anomalous chiral Hall current

$$J_{\text{diff}}^i = -D^0_{ij} \nabla_j \rho - (D^0_{\chi})_{ij} \nabla_j \rho_5,$$

$$D^0_{ij} = \frac{1}{2}(\sigma^0_{e_0} \delta_{ij} + D^0_H \epsilon_{ikj} B_k \mu), \quad (D^0_{\chi})_{ij} = \frac{1}{2}(\sigma^0_{a_\chi H} \epsilon_{ikj} E_k + D^0_H \epsilon_{ikj} B_k \mu_5).$$

• CMW ($\vec{q} \cdot \vec{B}$) and chiral Hall density wave (CHDW) ($\vec{q} \cdot (\vec{E} \times \vec{B})$)
Towards a self-consistent Chiral MHD

Charges and currents induce e/m fields of their own. The external electromagnetic fields $\vec{E}, \vec{B}$ have to be promoted into dynamical

$$\nabla \cdot \vec{E} = 4\pi \rho, \quad \nabla \times \vec{B} = \frac{1}{c} \left(4\pi \vec{J} + \partial_t \vec{E}\right), \quad \nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} = -\partial_t \vec{B}$$

Supplemented by the constitutive relations describing plasma medium effects

$$\vec{J} = \vec{J}[\rho, \rho_5, T, \vec{E}, \vec{B}]; \quad \vec{J}_5 = \vec{J}_5[\rho, \rho_5, T, \vec{E}, \vec{B}],$$

The constitutive relations are the ones where various hydrodynamic approximations are applied. A great deal of modelling is normally entering, such as truncated gradient expansions, weak field approximation, etc. It thus becomes mandatory to check if the original approximations made for the constitutive relations are consistent with the solution found. If not, the hydrodynamical model has to be revised.
Anomalous Hydro from Fluid-Gravity correspondence

\( U_V(1) \times U_A(1) \) Maxwell-Chern-Simons theory in the Schwarzschild-AdS\(_5\).

\[
\mathcal{L} = -\frac{1}{4} (F^V)_{MN}(F^V)^{MN} - \frac{1}{4} (F^a)_{MN}(F^a)^{MN} + \frac{\kappa \epsilon^{MNPQR}}{2\sqrt{-g}} \\
\times \left[ 3A_M(F^V)_{NP}(F^V)_{QR} + A_M(F^a)_{NP}(F^a)_{QR} \right],
\]

Chemical potentials:

\[
\mu = A_t(r = \infty) - A_t(r = 1) = \rho/2 - A_t(r = 1),
\]

\[
\mu_5 = A^a_t(r = \infty) - A^a_t(r = 1) = \rho_5/2 - A^a_t(r = 1)
\]

\[
\mu = \mu[\rho, \vec{E}, \vec{B}] \quad \mu_5 = \mu[\rho_5, \vec{E}, \vec{B}]
\]
I. Linear transport: weak fields

\[ \rho(x) = \bar{\rho} + \delta \rho(x), \quad \rho_5(x) = \bar{\rho}_5 + \delta \rho_5(x), \]

\[ \mu(x) = \bar{\mu} + \delta \mu(x), \quad \mu_5(x) = \bar{\mu}_5 + \delta \mu_5(x), \quad \bar{\mu} = \bar{\rho}/2, \quad \bar{\mu}_5 = \bar{\rho}_5/2 \]

Linear in \( \vec{E} \& \vec{B} \) and linear in \( \delta \rho \)

\[ J^t = \rho, \quad \tilde{J} = -D \vec{\nabla} \rho + \sigma_e E + \sigma_m \vec{\nabla} \times \vec{B} + \sigma_\chi \vec{B} \]

\[ J^t_5 = \rho_5, \quad J^i_5 = -D \vec{\nabla} \rho_5 + \sigma_a \vec{\nabla} \times \vec{B} + \sigma_\kappa \vec{B}. \quad \partial_\mu J_5^\mu = 0 \]


No linear chiral electric separation effect (CESE) in our model (\( \tilde{J}_5 \sim \vec{E} \))

Linear constitutive relations lead to a consistent \( \chi \)MHD as long as the field amplitudes remain weak
\( \sigma_m = 72 \kappa^2 \left( \bar{\mu}^2 + \bar{\mu}_5^2 \right) (2 \log 2 - 1) + i \omega \left[ \frac{1}{16} (2\pi - \pi^2 + 4 \log 2) + \mathcal{O} \left( \bar{\mu}^2 + \bar{\mu}_5^2 \right) \right] + \cdots, \)

\[ \sigma_m[q = 0] - \sigma_m[q = 0, \bar{\mu} = \bar{\mu}_5 = 0] \text{ is linear in } \kappa^2 (\bar{\mu}^2 + \bar{\mu}_5^2) \]

\[ \sigma_\chi = 12 \kappa \bar{\mu}_5 \left\{ 1 + i \omega \log 2 - \frac{1}{4} \omega^2 \log^2 2 - \frac{q^2}{24} \left[ \pi^2 - 1728 \kappa^2 (\bar{\mu}_5^2 + 3 \bar{\mu}^2) (\log 2 - 1)^2 \right] \right\} + \cdots, \]

\[ \sigma_\chi^0 \quad \text{A. Gynther, K. Landsteiner, F. Pena-Benitez, and A. Rebhan, (2011)} \]

\[ \sigma_\chi[q = 0] \text{ is linear in } \kappa \mu_5 \text{ and independent of } \mu \]

\[ \sigma_\kappa[\mu, \mu_5] = \sigma_\chi[\mu_5, \mu] \]

Plus tons of plots for arbitrary \( \omega, q \) and \( \mu, \mu_5 \)
$\text{Re}(\sigma_\chi), \kappa \bar{\mu}_5 = 0.25, \kappa \bar{\mu} = 0$

$\text{Im}(\sigma_\chi), \kappa \bar{\mu}_5 = 0.25, \kappa \bar{\mu} = 0$

$q = 0$

H.-U. Yee, (2009)
II. Constant fields: a) zeroth order in gradients

Fields of arbitrary strength. Linearisation in $\rho$ and $\rho_5$

\[
\vec{J}^{[0]} = \sigma^0_e \vec{E} + \sigma^0_\chi \kappa \rho_5 \vec{B} + \delta \sigma^0_\chi \kappa^2 (\vec{E} \cdot \vec{B}) \vec{B} + \sigma^0_{\chi H} \kappa^2 \rho \vec{B} \times \vec{E} + \sigma^0_{\chi e} \kappa^3 \rho_5 (\vec{B} \cdot \vec{E}) \vec{E},
\]

\[
\vec{J}^*_5^{[0]} = \sigma^0_\chi \kappa \rho \vec{B} + \sigma^0_{\chi H} \kappa^2 \rho_5 \vec{B} \times \vec{E} + \sigma^0_{\chi e} \kappa^3 \rho (\vec{B} \cdot \vec{E}) \vec{E} + \sigma^0_s \kappa^3 (\vec{E} \cdot \vec{B}) \vec{B} \times \vec{E},
\]

$\sigma^0_{\chi H}$ – chiral Hall effect;  $\sigma^0_{\chi e}$ – non-linear chiral electric effect /CESE;

$\delta \sigma^0_\chi$ – $\vec{E} \cdot \vec{B}$ -induced CME;  $\sigma_s$ – $\vec{E} \cdot \vec{B}$ -induced chiral Hall;

All the transport coefficients $\sigma_i = \sigma_i[\vec{E}^2, \vec{B}^2, \vec{E} \cdot \vec{B}]$

Because fields are generated by the currents, the above constitutive relations are not self-consistent
Electric Conductivity

Dependence on the angle between $\vec{E}$ and $\vec{B}$
CME Conductivity

\( \sigma_x^0 / (\sigma_x^0 [B = E = 0]) \)

- Graphs showing the relationship between \( \kappa E \) and \( \kappa B \) with \( \sigma_x^0 \) and \( (\kappa B)\sigma_x^0 \) as functions.

- Various plots illustrating different cases for \( \alpha = 0 \), \( \alpha = \pi/4 \), and \( \alpha = \pi/2 \).
II. Constant fields: b) first order in gradients

**Linear in \( \rho \) and \( \rho_5 \)**

\[
\mathbf{E} = 0
\]

\[
\mathbf{j}^{[1]} = -D_0 \vec{\nabla} \rho + \tau \kappa \partial_t \rho_5 \mathbf{B} + D^0_B \kappa^2 (\mathbf{B} \cdot \vec{\nabla} \rho) \mathbf{B},
\]

\[
\mathbf{j}_5^{[1]} = -D_0 \vec{\nabla} \rho_5 + \tau \kappa \partial_t \rho \mathbf{B} + D^0_B \kappa^2 (\mathbf{B} \cdot \vec{\nabla} \rho_5) \mathbf{B}
\]

\[
\mathbf{B} = 0
\]

\[
\mathbf{j}^{[1]} = -D_0 \vec{\nabla} \rho + \sigma^0_{\alpha \chi H} \mathbf{E} \times \vec{\nabla} \rho_5 + D^0_E \kappa^2 (\mathbf{E} \cdot \vec{\nabla} \rho) \mathbf{E},
\]

\[
\mathbf{j}_5^{[1]} = -D_0 \vec{\nabla} \rho_5 + \sigma^0_{\alpha \chi H} \mathbf{E} \times \vec{\nabla} \rho + D^0_E \kappa^2 (\mathbf{E} \cdot \vec{\nabla} \rho_5) \mathbf{E}
\]
Diffusion constant (weak field expansion)

\[
D_0 = \frac{1}{2} - 18(2 \log 2 - 1) \kappa^2 B^2 - \frac{3}{4} \pi^2 \kappa^2 E^2 + \cdots
\]

E = 0:
CMW and CHDW

Weak field expansion

\[
\omega = \pm \left[ 1 - 36(2 \log 2 - 1)\kappa^2 B^2 - \frac{3\pi^2}{2}\kappa^2 E^2 \right] 6\kappa (\vec{q} \cdot \vec{B}) \pm 9\pi^2 (\vec{E} \cdot \vec{B}) \kappa^3 (\vec{q} \cdot \vec{E}) \\
+ (36 \log 2)\kappa^2 (\vec{q} \cdot \vec{S}) - \left[ \frac{1}{2} + 18(1 - 2 \log 2)\kappa^2 B^2 - \frac{3\pi^2}{4}\kappa^2 E^2 \right] iq^2 \\
\pm \frac{9}{2} \log 2 \kappa (\vec{q} \cdot \vec{B}) q^2 - \frac{i}{8} q^4 \log 2 - i\frac{3}{4} \pi^2 \kappa^2 (\vec{q} \cdot \vec{E})^2 + i(36 \log 2)\kappa^2 (\vec{q} \cdot \vec{B})^2 + \cdots
\]

\[\vec{S} = \vec{E} \times \vec{B}\]

\[\omega \sim (\vec{q} \cdot \vec{S})\] - new gapless excitation - Chiral Hall Density Wave
**II. Constant weak fields: c) gradient resummation**

\[ \rho(x_\alpha) = \bar{\rho} + \delta\rho(x_\alpha), \quad \rho_5(x_\alpha) = \bar{\rho}_5 + \delta\rho_5(x_\alpha) \]

Linear in \( \vec{E} \& \vec{B} \) and linear in \( \delta\rho \)

\[
\delta \tilde{J}^{(1)(1)} = \sigma \chi \kappa \vec{B} \delta\rho_5 - \frac{1}{4} \mathcal{D}_H(\bar{\rho} \vec{B} \times \vec{\nabla} \delta\rho) - \frac{1}{4} \bar{\mathcal{D}}_H(\bar{\rho}_5 \vec{B} \times \vec{\nabla} \delta\rho_5) \\
- \frac{1}{2} \sigma_{a\chi H}(\vec{E} \times \vec{\nabla} \delta\rho_5) - \frac{1}{2} \bar{\sigma}_{a\chi H}(\vec{E} \times \vec{\nabla} \delta\rho) \\
+ \sigma_1 \kappa \left[ (\vec{B} \times \vec{\nabla}) \times \vec{\nabla} \right] \delta\rho + \sigma_2 \kappa \left[ (\vec{B} \times \vec{\nabla}) \times \vec{\nabla} \right] \delta\rho_5 \\
+ \sigma_3 \kappa \left[ (\vec{E} \times \vec{\nabla}) \times \vec{\nabla} \right] \delta\rho + \bar{\sigma}_3 \kappa \left[ (\vec{E} \times \vec{\nabla}) \times \vec{\nabla} \right] \delta\rho_5,
\]

\[
\delta \tilde{J}_5^{(1)(1)} = \sigma \chi \kappa \vec{B} \delta\rho - \frac{1}{4} \mathcal{D}_H(\bar{\rho} \vec{B} \times \vec{\nabla} \delta\rho_5) - \frac{1}{4} \bar{\mathcal{D}}_H(\bar{\rho}_5 \vec{B} \times \vec{\nabla} \delta\rho_5) \\
- \frac{1}{2} \sigma_{a\chi H}(\vec{E} \times \vec{\nabla} \delta\rho_5) - \frac{1}{2} \bar{\sigma}_{a\chi H}(\vec{E} \times \vec{\nabla} \delta\rho) \\
+ \sigma_1 \kappa \left[ (\vec{B} \times \vec{\nabla}) \times \vec{\nabla} \right] \delta\rho_5 + \sigma_2 \kappa \left[ (\vec{B} \times \vec{\nabla}) \times \vec{\nabla} \right] \delta\rho \\
+ \sigma_3 \kappa \left[ (\vec{E} \times \vec{\nabla}) \times \vec{\nabla} \right] \delta\rho_5 + \bar{\sigma}_3 \kappa \left[ (\vec{E} \times \vec{\nabla}) \times \vec{\nabla} \right] \delta\rho
\]
In the hydrodynamic limit $\omega, q \ll 1$, the TCFs are analytically computable:

$$\sigma_{\chi} = 6 + \frac{3}{2}i\omega (\pi + 2 \log 2) - \frac{1}{8} \left\{ \omega^2 \left[ \pi^2 + 6 \left( 4C + \log^2 2 \right) \right] + q^2 (12\pi - 24 \log 2) \right\} + \cdots,$$

$$\mathcal{D}_H = \kappa^2 \left\{ 72(3 \log 2 - 2) + i\omega 6 \left[ \pi (2\pi + 3 \log 2 - 6) + (9 \log 2 - 12) \log 2 \right] + \cdots \right\},$$

$$\tilde{\mathcal{D}}_H = \mathcal{D}_H [\mu \leftrightarrow \mu_5],$$

$$\sigma_{a\chi H} = \kappa \left\{ 6 \log 2 + i\omega \frac{1}{16} \left( 48C + 5\pi^2 \right) + \cdots \right\}, \quad \tilde{\sigma}_{a\chi H} = 0 + \cdots,$$

$$\sigma_1 = 162\kappa^2 \mu \mu_5 [6 + \log 2(5 \log 2 - 12)] + \cdots,$$

$$\sigma_2 = \frac{1}{8} (6\pi - \pi^2 - 12 \log 2) + 108\kappa^2 (\mu^2 + \mu_5^2) [6 + \log 2(5 \log 2 - 12)] + \cdots,$$

$$\sigma_3 = 9\kappa \mu \log^2 2 + \cdots, \quad \tilde{\sigma}_3 = \sigma_3 [\mu \leftrightarrow \mu_5].$$
$Re(\sigma_\chi)$

$Im(\sigma_\chi)$
Via inverse Fourier transform, the memory function

\[ \tilde{\sigma}_\chi(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \sigma_\chi(\omega, q = 0). \]

No instantaneous CME response!
CME response is delayed by a time of order temperature
Hall diffusion TCF

\[ \vec{J} \sim \mathcal{D}_H(\bar{\rho}\vec{B} \times \vec{\nabla}\delta\rho) \]
\[ \vec{J}_5 \sim \mathcal{D}_H(\bar{\rho}\vec{B} \times \vec{\nabla}\delta\rho_5) \]

Re(\mathcal{D}_H/\kappa^2), \kappa\bar{\mu}_5 = 0, \kappa\bar{\mu} = 0.25

Im(\mathcal{D}_H/\kappa^2), \kappa\bar{\mu}_5 = 0, \kappa\bar{\mu} = 0.25
Anomalous chiral Hall TCF

\[ \vec{J} \sim \sigma_{a\chi H}(\vec{E} \times \vec{\nabla} \delta \rho_5) \]

\[ \vec{J}_5 \sim \sigma_{a\chi H}(\vec{E} \times \vec{\nabla} \delta \rho) \]

\[ \text{Re}(\sigma_{a\chi H}/\kappa), \kappa\bar{\mu}_5 = 0.0625, \kappa\bar{\mu} = 0.0625 \]

\[ \text{Im}(\sigma_{a\chi H}/\kappa), \kappa\bar{\mu}_5 = 0.0625, \kappa\bar{\mu} = 0.0625 \]
Non-dissipative CMW modes

$$\omega = \pm \left( \bar{\sigma} \bar{\chi} - q^2 D \chi \right) \kappa \vec{B} \cdot \vec{q} - i \left( q^2 D - D_B (\kappa \vec{B} \cdot \vec{q})^2 \right)$$
Conclusions

- Memory function is an important ingredient of causal relativistic hydrodynamics. At large momenta, the effective viscosity, TCFs and many others are decreasing functions of both frequency and momentum. Fluid-gravity correspondence provides a calculational framework to rigorously address transports In QFT, including all order resummation. Unfortunately not in QCD ...

- All order dissipative terms of a weakly perturbed conformal fluid are fully accounted for by two shear viscosity functions $\eta(\omega, q^2)$ and $\zeta(\omega, q^2)$. An off-shell constitutive relation for $U(1)$ current consists of a momenta-dependent diffusion term and two conductivities. Certain universality between dissipative TCFs $\eta$ and $D$ is observed.

- We have re-examined transport coefficients induced by the chiral anomaly. We seem to be able to rediscover all known anomaly-induced effects within one and the same holographic model. We have got a zoo of new transport phenomena and corresponding transport coefficients (over 50 computed analytically)

- We have discovered a new dissipation-less CMW

- We propose to use generalised and more correct constitutive relations (memory functions and non-linear effects) for practical simulations of relativistic hydrodynamics and particularly of $\chi$MHD.