Maximal super Yang-Mills on spheres

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Based on:
JAM: arXiv:1512.06924
Anastasios Gorantis, Usman Naseer, JAM: arXiv:1711.05669
Nikolay Bobev, Pieter Bomans, Fridrik Gautason, Anton Nedelin, JAM: to appear

Holographic QCD
Nordita, 22 July 2019
Introduction

▶ We will discuss maximal SYM (MSYM) on $S^d$ for $d \leq 7$ [16 SUSYs]
▶ Why? (Theories are not renormalizable for $d > 4$):
  They have interesting UV completions.
    ▶ $d = 5$: 6d (2,0) theory compactified on $S^1$, $g^2_{YM} = R_6$.
    ▶ $d = 6$: 6d (1,1) LST
    ▶ $d = 7$: 7d “little m-theory”

We can also compute interesting quantities
▶ How do we know we can put SYM on spheres?: Available supergroup

$\mathcal{N} = 4$ in 4d: $SU(4|4)$
$\mathcal{N} = 2$ in 5d: $SU(4|1, 1)$ (noncompact $R$-symmetry group)
$\mathcal{N} = 2$ on $S^6$: $F(4) \supset SO(7) \times SU(1, 1)$
$\mathcal{N} = 1$ on $S^7$: $OSp(8|2, \mathbb{R})$

But not:
$\mathcal{N} = 1$ on $S^d$: $d > 7$
Introduction

- We will consider these theories as well as $\mathcal{N} = 8$ in 3D.
- We can also consider noninteger dimensions which will be useful for regularization.
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Supersymmetric gauge theories on spheres can be localized:

- $\mathcal{N} = 4$ and $\mathcal{N} = 2^* \text{ SYM in 4d}$ Pestun
- $\mathcal{N} = 2$ and higher SYM/CS in 3d Kapustin, Willett, Yakov; Jafferis
- $\mathcal{N} = 1, 2 \text{ SYM in 5d}$: Källén, Zabzine; Källén, Qiu, Zabzine; Kim, Kim
- $(2, 2) \text{ SYM in 2d}$: Benini and Cremonisi; Daroud et al.
- $\mathcal{N} = 2 \text{ 6d and } \mathcal{N} = 1 \text{ 7d SYM}$ Zabzine, JAM

Localization of MSYM can be generalized (perturbatively) to noninteger $d$ JAM; Gorantis, Naseer, JAM
Introduction

Question

Except for \( d = 4 \), these theories are not conformal. Can localization results for \( d \neq 4 \) be matched with results in supergravity?

Successful example: 4D \( N = 2^* \): Russo, Zarembo, Buchel; Bobev, Elvang, Freedman, Pufu
Review a generalized version of Pestun’s construction and perturbative localization for MSYM on $S^d$ for $d \leq 7$

Free energies and BPS Wilson loops for MSYM in $d < 6$
(Results for $d = 3$ require analytic continuation)

The case for $S^7$

Comparison to supergravity results sourced by spherical E branes
(Focus on $S^7$ case).

Concluding remarks
Dimensional reduction of MSYM

- **10-dimensional flat-space Lagrangian:** Brink, Scherk & Schwarz

\[ \mathcal{L} = -\frac{1}{g_{10}^2} \text{Tr} \left( \frac{1}{2} F_{MN} F^{MN} - \Psi \bar{\Phi} \Psi \right). \]

- **Action is invariant under the supersymmetry transformations**

\[
\begin{align*}
\delta_{\epsilon} A_M &= \epsilon^\alpha \Gamma_{M\alpha\beta} \Psi^\beta, \quad M = 0, \ldots 9 \\
\delta_{\epsilon} \Psi^\alpha &= \frac{1}{2} \Gamma^{MN\alpha}_{\beta} F_{MN} \epsilon^\beta, \quad \alpha, \beta = 1, \ldots 16
\end{align*}
\]

\(\epsilon^\alpha\) are bosonic real chiral constant spinors; (16 independent SUSYs)

- **Dimensionally reduce to \(d\)-dimensional Euclidean gauge theory.**

\[ A_\mu, \quad \mu = 1, \ldots, d \quad \phi_I \equiv A_I, \quad I = 0, d+1, \ldots 9. \]

- **Derivatives along compactified directions are zero:**

\[ F_{\mu I} = [D_\mu, \phi_I] \quad F_{IJ} = [\phi_I, \phi_J]. \]

- **Scalars transform under vector rep. of \(SO(1, 9-d)\) \(R\)-symmetry in flat Euclidean space.** \(\phi_0\) has wrong-sign kinetic term.

- **\(d\)-dimensional coupling:**

\[ g_{YM}^2 = g_{10}^2 / V_{10-d}. \]
The theory on spheres  Blau ’00, Zabzine and JM ’15

- Put theory on $S^d$ with radius $r$.
- $d = 4$: MSYM is superconformal, $\Longrightarrow$ conformal mass term

\[ S_{\phi\phi} = \frac{1}{g_{YM}^2} \int d^4x \sqrt{-g} \left( \frac{2}{r^2} \text{Tr}\phi_1\phi^1 \right) \]

- $d \neq 4$: not superconformal, but we need a similar term as well as other terms to preserve supersymmetry.
Lagrangian on $S^d$

- SUSY transfs. need to be modified

\[ \delta_\epsilon A_M = \epsilon \Gamma_M \Psi \]
\[ \delta_\epsilon \Psi = \frac{1}{2} \Gamma^{MN} F_{MN} \epsilon + \frac{\alpha_I}{2} \Gamma^{\mu I} \phi_I \nabla_\mu \epsilon, \quad \nabla_\mu \epsilon = \frac{1}{2r} \Gamma_\mu \Gamma^0 \Gamma^8 \Gamma^9 \epsilon \]

\[ \alpha_A = \frac{4(d-3)}{d}, \quad A = 8, 9, 0, \quad \alpha_i = \frac{4}{d}, \quad i = d + 1, \ldots 7 \]

- Complete maximally SUSY Lagrangian (up to d=7):

\[ \mathcal{L}_{ss} = \frac{1}{g_{YM}^2} \text{Tr} \left[ \left( \frac{1}{2} F_{MN} F^{MN} - \Psi \bar{\partial} \Psi \right. \right. \]
\[ \left. \left. + \frac{2(d - 3)}{r^2} \text{Tr} \phi^A \phi_A + \frac{(d - 2)}{r^2} \text{Tr} \phi^i \phi_i + \frac{(d - 4)}{2r} \Psi \wedge \Psi \right. \]
\[ \left. - \frac{4}{r} (d - 4) \text{Tr} (\phi^0 [\phi^8, \phi^9]) \right] \]

- Preserves 16 susys but $R$-symmetry explicitly broken ($d \neq 4, 7$):

\[ SO(1, 9 - d) \rightarrow SO(1, 2) \times SO(7 - d) \]
Localization

► In the large $N$-limit (where we can ignore instantons) the theory can be localized onto constant $\phi_0$, with all other fields zero.

\[ \mathcal{L}_{fp} = \frac{1}{g_{YM}^2} \frac{(d-1)(d-3)}{r^2} \text{Tr}(\phi_0 \phi_0), \quad \nabla_\mu \phi_0 = 0. \]

► Wick rotate $\phi_0 \rightarrow i\phi_0$ and define dimensionless variable: $\sigma = r\phi_0$.

\[ S_E = V_d \mathcal{L}_E = \frac{r^{d-4}(d-1)(d-3)S_d}{g_{YM}^2} \text{Tr}\sigma^2 = \frac{4\pi^2 r^{d-4}S_{d-4}}{g_{YM}^2} \text{Tr}\sigma^2 \]

Zabzine, JAM 2015
LL determinants (16 susys) (JAM ’15; Gorantis, Naseer & JAM ’17)

- General $d$ determinant factor:

$$
\prod_{\gamma>0} \frac{1}{\langle \gamma, \sigma \rangle^2} \prod_{n=0}^{\infty} \left( \frac{n^2 + \langle \gamma, \sigma \rangle^2}{(n + d - 3)^2 + \langle \gamma, \sigma \rangle^2} \right)^{\frac{\Gamma(n+d-3)}{\Gamma(n+1)\Gamma(d-3)}}
$$

This is divergent and needs to be regularized.

- Perturbative partition function in terms of eigenvalues of $\sigma$

$$
Z = \int \prod_{i=1}^{N} d\sigma_i e^{-\frac{4\pi^2 r^{d-4} s_{YM}^{d-4}}{g_{YM}} \sum_i \sigma_i^2 \prod_{i<j} \prod_{n=0}^{\infty} \left( \frac{n^2 + (\sigma_i - \sigma_j)^2}{(n + d - 3)^2 + (\sigma_i - \sigma_j)^2} \right)^{\frac{\Gamma(n+d-3)}{\Gamma(n+1)\Gamma(d-3)}}}
$$

- In the large $N$ limit we can solve by saddle point,
Eigenvalue equations of motion

- Force equation for eigenvalues:

\[
\frac{C_1 N}{\lambda} \sigma_i = \sum_{j \neq i} G(\sigma_{ij}), \quad \lambda \equiv g_{YM}^2 N r^{4-d}
\]

\[
C_1 \equiv \frac{16\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d-3}{2}\right)},
\]

\[
G(\sigma) \equiv -i \Gamma(4 - d) \times \left( \frac{\Gamma(-i \sigma)}{\Gamma(4 - d - i \sigma)} - \frac{\Gamma(i \sigma)}{\Gamma(4 - d + i \sigma)} - \frac{\Gamma(d - 3 - i \sigma)}{\Gamma(1 - i \sigma)} + \frac{\Gamma(d - 3 + i \sigma)}{\Gamma(1 + i \sigma)} \right).
\]
For small separations, $|\sigma_{ij}| \ll 1$ kernel has weak coupling behavior:

$$G(\sigma_{ij}) \approx \frac{2}{\sigma_{ij}}.$$  

For large separations, $|\sigma_{ij}| \gg 1$ kernel has strong coupling behavior:

$$G(\sigma_{ij}) \approx C_2 |\sigma_{ij}|^{d-5} \text{sign}(\sigma_{ij})$$

$$C_2 = 2(d-3)\Gamma(5-d) \sin \left( \frac{\pi(d-3)}{2} \right)$$
Free energy for maximal SYM \((d < 6)\)

- We are interested in \(\lambda \gg 1 \iff \text{weak eigenvalue central potential, repulsive force terms in } G(\sigma_{ij}) \text{ push the eigenvalues far apart.}\)
- We write the saddle point equation as an integral equation

\[
\frac{C_1}{\lambda} \sigma = C_2 \int_{-b}^{b} d\sigma' \rho(\sigma') |\sigma - \sigma'|^{d-5} \text{sign}(\sigma - \sigma') .
\]

- Scaling: \(\sigma \sim \lambda^{\frac{1}{6-d}} \Rightarrow F \sim \lambda^{\frac{d-4}{6-d}} N^2 \quad \text{JAM 2015}\)
- We can solve the integral equation to find the density

\[
\rho(\sigma) = K (b^2 - \sigma^2)^{\frac{5-d}{2}} \quad -b \leq \sigma \leq b , \quad K = \frac{\Gamma(4 - \frac{d}{2})}{b^{6-d} \pi^{1/2} \Gamma(\frac{7-d}{2})}
\]

\[
\rho(\sigma) = 0 \quad |\sigma| > b
\]

\[
b = \left( \frac{\lambda \sin \frac{\pi(d-3)}{2} \Gamma(5 - d) \Gamma(\frac{d-3}{2}) \Gamma(\frac{d-1}{2}) \Gamma(4 - \frac{d}{2})}{2\pi^{\frac{d+2}{2}}} \right)^{\frac{1}{6-d}}.
\]
Eigenvalue distribution \((d < 6)\)

- **Cases:**

\[
d = 4 : \quad \rho(\sigma) = \frac{8\pi}{\lambda} \left( \frac{\lambda}{4\pi^2} - \sigma^2 \right)^{1/2}
\]

\[
d = 5 : \quad \rho(\sigma) = \frac{4\pi^2}{\lambda}; \quad b = \frac{\lambda}{8\pi^2}
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\[
d = 3 : \quad \mathcal{Z} = \int \prod_{i=1}^{N} d\sigma_i \ 1
\]

14 / 24
Eigenvalue distribution ($d < 6$)

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\]

\[
d = 3 : \quad \mathcal{Z} = \int \prod_{i=1}^{N} d\sigma_i \quad ???
\]

\[
d \to 3^+ : \quad \rho(\sigma) = \frac{2\pi}{\lambda} \left( \frac{1}{4} \left( \frac{3\lambda}{\pi} \right)^{2/3} - \sigma^2 \right)^{1/3}
\]
Free energy for maximal SYM ($d < 6$)

- Substitute $\rho(\sigma)$ into the action to find the free energy in terms of $d$:

$$F = -\frac{16\pi^{d+1/2}(6-d)N^2}{\lambda \Gamma(\frac{d-3}{2})(8-d)(d-4)} \left( \lambda \cos \frac{\pi d}{2} \frac{\Gamma(5-d)\Gamma(\frac{d-3}{2})\Gamma(\frac{d-1}{2})\Gamma(4-d)}{2\pi^{\frac{d+2}{2}}} \right)^{\frac{2}{6-d}}$$

- Cases:

\[d = 4-\epsilon:\quad F_4 \approx -\frac{4\pi^2 N^2}{\lambda \epsilon} \left( \frac{\lambda}{4\pi^2} \right)^{1+\epsilon/2} \approx -\frac{N^2}{\epsilon} - \frac{N^2}{2} \log \lambda\]

\[d = 5:\quad F_5 = -\frac{8\pi^3 N^2}{3\lambda} \left( \frac{\lambda}{8\pi^2} \right)^2 = -\frac{\lambda N^2}{24\pi}\]

\[d \rightarrow 3^+: \quad F_3 = 0\]

\[d = 6-\epsilon:\quad F_6 = -\frac{8\pi^3 \epsilon N^2}{\lambda} \left( \frac{3\lambda}{16\pi^3 \epsilon} \right)^{2/\epsilon} \rightarrow -\infty\]
BPS Wilson loops

- BPS Wilson loop is along the equator of $S^d$

$$
\langle W \rangle = \text{Tr} \left( P e^{i \oint ds \cdot \phi_0} \right) \approx \int_{-b}^{b} d\sigma \rho(\sigma) e^{2\pi \sigma}.
$$

- Using the previous distribution $\rho(\sigma)$

$$
\langle W \rangle = (\pi b)^{\frac{d-6}{2}} \Gamma \left( \frac{8-d}{2} \right) I_{\frac{6-d}{2}} (2\pi b)
$$

- Cases:

  - $d = 4$ :
    $$
    \langle W \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \quad \text{ESZ '00}
    $$
  
  - $d = 5$ :
    $$
    \langle W \rangle = \frac{4\pi}{\lambda} \sinh \left( \frac{\lambda}{4\pi} \right)
    $$
  
  - $d \to 3_+$ :
    $$
    \langle W \rangle = \frac{1}{\pi^2 \lambda} (\xi \cosh(\xi) - \sinh(\xi)) , \quad \xi = (3\pi^2 \lambda)^{1/3}
    $$
$d = 7$

Central force ($\lambda \gg 1$)

Eigenvalue force

Central force ($\lambda \ll 1$)

$\lambda = \infty$

Eigenvalue distribution (solved numerically)

Eigenvalues are not widely separated

$\rightarrow$

Can't trust the wide separation approximation
$d = 7$

- **Central force ($\lambda \gg 1$)**
- **Eigenvalue force**
- **Central force ($\lambda \ll 1$)**

$\lambda = \infty$ eigenvalue distribution (solved numerically)

- Eigenvalues are not widely separated $\rightarrow$ Can’t trust the wide separation approximation
Let’s use the large separation approximation anyway:

► **Eigenvalue distribution**: $d \to 7$

$$b = -\frac{4\pi^3}{\lambda}, \quad \rho(\sigma) = \frac{1}{2}(\delta(\sigma + b) + \delta(\sigma - b))$$

► **Free energy**: $F_7 = \frac{(2\pi)^{10}N^2}{24\lambda^3}$

► **Wilson loop**: $\langle W \rangle_7 = \cosh \left( \frac{8\pi^4}{\lambda} \right)$
$d = 7$ (continued)

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- We should take $\lambda < 0$, such that $-\lambda^{-1} \gg 1$
Comparison to supergravity

- Supergravity duals to SYM in flat space were first considered by Itzhaki, Maldacena, Sonnenschein and Yankielowicz ’98
- Bobev, Bomans and Gautason found supergravity solutions sourced by spherical branes ’18
- We will concentrate on the $S^7$ case.

Some remarks:

- The dual string theory should arise from 7 dimensional Euclidean branes: “E branes” Hull 1998
- T dualize the time direction in IIB $\Rightarrow$ IIA $^*$ with (1,9) signature
- All $D_p$ branes in IIB become $E_p$ branes in IIA $^*$
- $E_p$ branes have imaginary tensions e.g. $E_1$ brane is a space-like world line $\Rightarrow$ tachyonic mass
- $E_p$ branes are BPS $\Rightarrow$ imaginary charges $\Rightarrow$ source imaginary fluxes
- In strong coupling limit of IIA $^*$, $E_1$ branes are Kaluza-Klein modes on the M theory circle. Imaginary masses $\Rightarrow$ time-like circle $\Rightarrow$ (2,9) signature.

"M $^*$ theory"
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    Imaginary masses $\Rightarrow$ time-like circle $\Rightarrow$ (2,9) signature.

“M* theory”
Comparison to supergravity

BBG argued that the supergravity dual for SYM on $S^7$ is 11D sugra on $S^7 \times H^{2,2}/\mathbb{Z}_N$, with imaginary flux $F_4$ through $H^{2,2}/\mathbb{Z}_N$. We will assume that $g_{YM}^2 < 0$, net effect changes the sign of the $F_4$ flux.

$$ds^2 = \frac{r^2}{4} (ds_4^2 + 4d\Omega_7^2)$$
$$ds_4^2 = d\rho^2 - \frac{\sinh^2 \rho}{4} (dt^2 - \cosh^2 t d\psi^2 + (N^{-1}d\omega - \sinh t d\psi)^2)$$

- $\omega$ is the direction along the time-like M* theory circle
- $d\Omega_2 = -dt^2 + \cosh^2 t d\psi^2$ has $SO(1, 2)$ isometry $\Rightarrow$ R-symmetry
- Locally there is an $SO(2, 3)$ isometry
- $ds_4^2$ has a conical singularity at $\rho = 0$ if $N \neq 1$. 
Wilson loop in supergravity

- The holographic dual for a BPS Wilson loop on the $S^7$ equator is an M2 brane with legs along the $S^7$ equator, and $\omega$ and $\rho$ on $H_{2,2}/\mathbb{Z}_N$.
- The brane is fixed along $t$ and $\psi$.
- Holographic dual for the Wilson loop:

\[
\langle W \rangle = e^{-S_{M2}}
\]

\[
S_{M2} = T_2 \int d^3x \sqrt{-g} \quad T_2 = \frac{2\pi}{(2\pi\ell_{11})^3}
\]

\[
\ell_{11} = (\frac{-g_s}{\ell_s})^{1/3} \ell_s \quad 2g_s = g_{YM}^2 (2\pi)^{-4} \ell_s^{-3}
\]

- M2 brane action is divergent $\Rightarrow$ include cutoff factor:

\[
\int d^3x \sqrt{g} = (2\pi r)^{\frac{4\pi}{N}} \times \frac{r^2}{4} \int_0^{\rho_0} d\rho \sinh \rho = \frac{2\pi^2 r^3}{N} (\cosh \rho_0 - 1)
\]

Standard regularization including a GH term removes $e^{\rho_0}$ divergence:

\[
\Rightarrow S_{M2} = \frac{8\pi^4}{\lambda}, \quad \Rightarrow \langle W \rangle_{SG} \sim \exp \left(-\frac{8\pi^4}{\lambda} \right) \checkmark
\]
The free energy from supergravity

- 11d bosonic action with imaginary flux

\[ S = \frac{1}{16\pi G} \int d^{11}x \sqrt{g} \left( R^{(7)} - 2\Lambda^{(7)} \right) = \frac{1}{16\pi G} \int d^{11}x \sqrt{g} \frac{12}{r^2} \]

- \( S \) is real with a (2,9) signature. We only Wick rotate \( t \rightarrow -i\tau \)

\[ ds_4^2 \rightarrow d\rho^2 + \frac{\sinh^2 \rho}{4} \left( d\tau^2 + \cos^2 \tau d\psi^2 - (N^{-1} d\omega + i \sin \tau d\psi)^2 \right) \]

- Even though the metric is complex \( \text{det } g \) is real.

- Compute “Euclidean action”

\[ S_E = -\frac{1}{16\pi G} \int d^{11}x \sqrt{-g} \frac{12}{r^2} = -\frac{\pi^3 r^7}{48 G} \left( \frac{12}{r^2} \right) \frac{\pi^2 r^4}{8N} \int_0^{\rho_0} d\rho \sinh^3 \rho \]

\[ \int_0^{\rho_0} d\rho \sinh^3 \rho = \frac{1}{3} \cosh^3 \rho_0 - \cosh \rho_0 + \frac{2}{3} \rightarrow \frac{2}{3} \]

- AdS/CFT dictionary: \( G = 16\pi^7 \ell_{11}^9 \)

\[ F_{SG} = S_E = \frac{(2\pi)^{10} N^2}{24\lambda^3} \]

\( \checkmark \)
Concluding remarks

- We can match the localization results for $S^7$ with supergravity if we assume that $-\lambda^{-1} \gg 1$. $r/\ell_{11} \sim N^{1/3}(-\lambda)^{-1/3}$, so there is small curvature.
Concluding remarks

- We can match the localization results for $S^7$ with supergravity if we assume that $-\lambda^{-1} \gg 1$. $r/\ell_{11} \sim N^{1/3}(-\lambda)^{-1/3}$, so there is small curvature.
- Perhaps more clear with IIA

$$
\begin{align*}
 ds_{10}^2 & \sim \left( \frac{\ell_s \sinh \rho}{(-\lambda)} \right) \left( d\rho^2 + 4 d\Omega_7^2 + \frac{1}{4} \sinh^2 \rho \, d\Omega_2^2 \right) \\
 e^{2\Phi} & \sim \frac{\sinh^3 \rho}{(-\lambda) N^2}
\end{align*}
$$

Valid if $(-\lambda)^{1/3} \ll \sinh \rho \ll (-\lambda)^{1/3} N^{2/3}$. Since the main contribution to the regularized integrals comes from near $\rho = 0$, we need to assume that $-\lambda \ll 1$
While we did not show it, we can also reproduce free energies and Wilson loops for other values of $d$ in supergravity (modulo some annoying numerical factors).

- $d = 3$: $\log \langle W \rangle_{SG} = \frac{3}{2} \log \langle W \rangle_{Loc}$
- $d = 2$: $\log \langle W \rangle_{SG} = 2^{1/4} \log \langle W \rangle_{Loc}$

Puzzle about 3d $N = 8$ SYM. The lore is that in the IR it should flow to an SCFT with 32 susys (ABJM $k = 1$) SYM: $\log \langle W \rangle \sim \lambda^{1/3} = (g_{YM}^2 r)^{1/3} N^{1/3}$, ABJM: $\log \langle W \rangle \sim N^{1/2}$

Possible crossover behavior when $\lambda \sim N^{3/2}$, but still not clear because of 0 coefficient in $F$. Possible order of limits problem.
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Puzzle about 3d $\mathcal{N} = 8$ SYM. The lore is that in the IR it should flow to an SCFT with 32 susys (ABJM $k = 1$)

- SYM: $\log \langle W \rangle \sim \lambda^{1/3} = (g_{YM}^2 r)^{1/3} N^{1/3}$, ABJM: $\log \langle W \rangle \sim N^{1/2}$
- SYM $F \sim 0 \times \lambda^{-1/3} N^2$, ABJM: $F \sim N^{3/2}$

Possible crossover behavior when $\lambda \sim N^{3/2}$, but still not clear because of 0 coefficient in $F$. Possible order of limits problem.
Thanks!
Other stuff
Free energy for maximal SYM (6d)

Go back to saddle point equation and set \( d = 6 - \epsilon \):

\[
\frac{32\pi^3}{\lambda} N \sigma_i = \frac{6}{\epsilon} \sum_{j \neq i} (\sigma_i - \sigma_j) \left(1 - \frac{\epsilon}{2} \log(\sigma_i - \sigma_j)^2\right)
\]

\[
= \frac{6}{\epsilon} N \sigma_i - 3 \sum_{j \neq i} (\sigma_i - \sigma_j) \log(\sigma_i - \sigma_j)^2.
\]

1st term on the rhs can be absorbed into coupling. Define \( \lambda_b, \lambda_r \)

\[
\frac{1}{\lambda_b} = \frac{3}{16\pi^3 \epsilon} + \frac{1}{\lambda_r}.
\]

In terms of \( \lambda_r \)

\[
F_6 = -\frac{3}{2} N^2 \left(1 - \frac{16\pi^6 \epsilon}{3\lambda_r}\right)^{2/\epsilon} = -\frac{3}{2} N^2 \exp \left(-\frac{32\pi^6}{3\lambda_r}\right).
\]

\[
\langle W \rangle \sim \exp \left(\exp \left(-\frac{32\pi^6}{3\lambda_r}\right)\right)
\]

Relation to (1,1) little string theory?

Little string tension: \( T = \frac{4\pi^2}{g_{YM}^2} \Rightarrow F_6 \sim -N^2 \exp(-C T r^2/N) \)
Lump at end of 7d eigenvalue density
Conformal Killing spinors

- No covariantly constant spinors on the sphere
- There are conformal Killing spinors (CKS)

\[ \nabla_\mu \epsilon^\alpha = \tilde{\Gamma}_\mu^{\alpha\beta} \tilde{\epsilon}_\beta, \quad \nabla_\mu \tilde{\epsilon}_\alpha = -\frac{1}{4r^2} \Gamma_{\mu\alpha\beta}\epsilon^\beta. \]

\( \tilde{\epsilon}_\alpha \) has opposite chirality to \( \epsilon^\alpha \).

- 32 independent solutions for \( d \leq 10 \):
- Reduce to 16 spinors by further imposing

\[ \tilde{\epsilon} = \beta \Lambda \epsilon \quad (\beta \equiv \frac{1}{2r}), \quad \tilde{\Gamma}^\mu \Lambda = -\tilde{\Lambda} \Gamma^\mu \quad \tilde{\Lambda} \Lambda = 1 \]

If \( d \neq 4 \) also need \( \Lambda^T = -\Lambda \quad \Rightarrow \quad \Lambda = \Gamma^8 \tilde{\Gamma}^9 \Gamma^0 \)

- This construction can be used for spheres up to \( d = 7 \).
Off-shell Lagrangian

- Look for the Lagrangian invariant under the SUSY transformations

\[
\begin{align*}
\delta_{\epsilon} A_M &= \epsilon \Gamma_M \psi, \\
\delta_{\epsilon} \psi &= \frac{1}{2} \Gamma^{MN} F_{MN} \epsilon + \frac{\alpha I}{2} \Gamma^{\mu I} \phi_I \nabla_\mu \epsilon + K^m \nu_m \\
\delta_{\epsilon} K^m &= -\nu^m (\bar{\psi} \gamma^5 \psi - (d - 4) \beta \Lambda \psi) \quad m = 1 \ldots 7
\end{align*}
\]

- Off-shell Lagrangian

\[
L_{\text{off}} = -\frac{1}{g_{YM}^2} \text{Tr} \left[ \left( \frac{1}{2} F_{MN} F^{MN} - \psi \gamma^5 \psi \right. \right.
\left. + \frac{2(d - 3)}{r^2} \text{Tr} \phi_A \phi_A + \frac{(d - 2)}{r^2} \text{Tr} \phi_i \phi_i + \frac{(d - 4)}{2r} \psi \Lambda \psi
- \frac{4}{r} (d - 4) \text{Tr} (\phi^0 [\phi^8, \phi^9]) - K^m K_m \right] \]

Localization

- Localizing the (off-shell) action ⇒ Modify the path integral to

\[ Z = \int \mathcal{D}\Phi e^{i S - tQV}, \]

where \( Q \) is a fermionic symmetry generator. \( QV \) is positive definite.

- Take \( t \to \infty \) so fields localize onto fixed loci of \( V \) under \( Q \).

\[ Z = \sum_{k \in \text{fixed loci}} \int \mathcal{D}\Phi_0 e^{i S_k \text{Det}_k} \]

- Positive definite \( QV \) requires Wick rotation \( \phi_0 \to i \phi_0, K^m \to i K^m \)

- Fixed-point locus (zero instanton sector with \( A_\mu = 0 \))

\[ K^m = -2\beta(d - 3)\phi_0 (\nu_m \Lambda \epsilon), \quad \nabla_\mu \phi_0 = 0 \quad \phi_J = 0 \quad J \neq 0. \]
EOM from imaginary flux

- Start with 11d bosonic action

\[
S = \frac{1}{16\pi G} \int d^{11}x \sqrt{\text{det} g} \left( R^{(4)} + R^{(7)} + \frac{1}{2} |F|^2 \right), \quad |F|^2 \equiv -F_{\mu_1...\mu_4} F^{\mu_1...\mu_4}
\]

- Solving the eom’s we find

\[
R^{(4)} = -\frac{4}{3} |F|^2, \quad R^{(7)} = \frac{7}{6} |F|^2
\]

\[
|F|^2 = \frac{9}{L_4^2}, \quad L_7 = 2L_4 = r
\]

- Substitute back into the action

\[
S = \frac{1}{16\pi G} \int d^{11}x \sqrt{g} \left( \frac{1}{3} \right) |F|^2 = \frac{1}{16\pi G} \int d^{11}x \sqrt{g} \frac{12}{r^2}.
\]
**LL determinants (8 susys)** (JAM ’15; Gorantis, JAM, & Naseer ’17)

Vector multiplet:

\[
\frac{\text{Det}_{f,v}}{\text{Det}_{b,v}} = \prod_{\gamma} \prod_{n=1}^{\infty} (n + i\langle \gamma, \phi_0 \rangle)^{\frac{\Gamma(n+d-2)}{\Gamma(n+1)\Gamma(d-2)}} \prod_{n=0}^{\infty} (n + d - 2 + i\langle \gamma, \phi_0 \rangle)^{\frac{\Gamma(n+d-2)}{\Gamma(n+1)\Gamma(d-2)}}
\]

Hypermultiplet \((\mu = m r)\):

\[
\frac{\text{Det}_{f,h}}{\text{Det}_{b,h}} = \prod_{\gamma} \prod_{n=0}^{\infty} \left[ \left( n + i\langle \gamma, \sigma \rangle + i\mu + \frac{d-2}{2} \right) \left( n - i\langle \gamma, \sigma \rangle - i\mu + \frac{d-2}{2} \right) \right]^{-\frac{\Gamma(n+d-2)}{\Gamma(n+1)\Gamma(d-2)}}
\]

- \[^d \leq 5: \text{combine a vector multiplet with an adjoint hyper with mass } \mu = i(d - 4)/2 \text{ to give 16 supercharges.}\]
  - For general \(d\) (after shifting some \(n\)) the combined determinant factor is (including the Vandermonde determinant)

\[
\prod_{\gamma > 0} \prod_{n=0}^{\infty} \left[ \left( \frac{n^2 + \langle \gamma, \sigma \rangle^2}{(n + d - 3)^2 + \langle \gamma, \sigma \rangle^2} \right) \right]^{\frac{\Gamma(n+d-3)}{\Gamma(n+1)\Gamma(d-3)}}
\]

- Analytically continuing to \(d > 5\) result agrees with \(d = 6\) and \(d = 7\) cases. JAM, Zabzine ’15