Axion holographic RG flows and the dynamics of topological densities

Francesco Nitti

APC, U. Paris Diderot

Holographic QCD, Nordita, 22-07-2019

Work with Y. Hamada, Elias Kiritsis, Lukas Witkowski

Work in progress with E. Kiritsis
Topological charge at large-$N$

\[ \mathcal{L}_{YM} = \frac{1}{4g^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{16\pi^2} \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \]
Topological charge at large-$N$

\[ \mathcal{L}_{YM} = N \left[ \frac{1}{4\lambda} Tr F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{16\pi^2 N} Tr F_{\mu\nu} \tilde{F}^{\mu\nu} \right], \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \]

Large-$N$ limit: keep $\lambda \equiv g^2 N$ finite.

\[ \mathcal{L}_{YM} = NL[\lambda, \theta/N] \]
Topological charge at large-$N$

\[
\mathcal{L}_{YM} = N \left[ \frac{1}{4\lambda} Tr F_{\mu\nu} F^{\mu\nu} + \frac{\theta}{16\pi^2 N} Tr F_{\mu\nu} \tilde{F}^{\mu\nu} \right], \quad \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}
\]

Large-$N$ limit: keep $\lambda \equiv g^2 N$ finite.

\[
\mathcal{L}_{YM} = NL[\lambda, \theta/N]
\]

- Quantities obtained from $L[\lambda, \theta/N]$ have good large-$N$ limit
- $\theta \in [0, 2\pi] \Rightarrow$ the contribution of the topological term to glue dynamics is suppressed at large $N$. E.g. for small $\theta$ (Witten):

\[
\mathcal{E}(\lambda, \theta) \approx N^2 \mathcal{E}(\lambda, 0) + \frac{1}{2} \chi \theta^2, \quad \chi = \mathcal{E}''(\lambda, 0)
\]
Holographic Setup

Bottom-up 5-dimensional model with Axion

\[ S = M_p^3 \int d^4x dr \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - V(\phi) + Y(\phi) \frac{(\partial a)^2}{2} \right] \]

- \( \phi \Leftrightarrow \) relevant operator \( O \) of UV-dimension \( \Delta \)
- IHQCD: \( \lambda = e^\phi = \) (running) ’t Hooft coupling \( \Leftrightarrow Tr F^2(x) \)
Holographic Setup

Bottom-up 5-dimensional model with Axion

\[ S = M_p^3 \int d^4x dr \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - V(\phi) + Y(\phi) \frac{(\partial a)^2}{2} \right] \]

- \( \phi \Leftrightarrow \) relevant operator \( O \) of UV-dimension \( \Delta \)
- IHQCD: \( \lambda = e^\phi = \) (running) ’t Hooft coupling \( \Leftrightarrow Tr F^2(x) \)

\[ a(x, r) \Leftrightarrow q(x) = \frac{1}{16\pi^2} Tr F \tilde{F}(x) \]
Holographic Setup

Bottom-up 5-dimensional model with Axion

\[ S = M_p^3 \int d^4x dr \sqrt{-g} \left[ R - \frac{1}{2} (\partial \phi)^2 - V(\phi) + Y(\phi) \frac{(\partial a)^2}{2} \right] \]

- \( \phi \Leftrightarrow \) relevant operator \( O \) of UV-dimension \( \Delta \)
- IHQCD: \( \lambda = e^\phi = \) (running) ’t Hooft coupling \( \Leftrightarrow Tr F^2(x) \)
- \( a(x, r) \Leftrightarrow q(x) = \frac{1}{16\pi^2} Tr F \tilde{F}(x) \)

- Exact shift symmetry in the large-\( N \) limit \( \Rightarrow \) No potential for \( a \).
- \( V(\lambda), Y(\lambda) \) to be fixed phenomenologically.
Outline

- **PART I**
  Axionic RG flow solutions in bottom-up HQCD
  1905.03663 Y. Hamada, E. Kiritsis, FN, L. Witkowski

- **PART II**
  Role of contact terms in holographic correlators
  FN, E. Kiritsis, in progress
PART I

Axion RG flows

- Bulk evolution of $a(r)$ may be understood as non-perturbative running of $\theta$.
- Study at exact holographic RG-flow solutions of Einstein-Axion-Dilaton (exception to standard multi-field superpotential formalims)
- Potentially interesting for pheno (relaxion, strong CP).

1905.03663 Y. Hamada, Elias Kiritsis, FN, Lukas Witkowski
Axionic domain-wall solutions

\[ ds^2 = du^2 + e^{2A(u)} dx^\mu dx_\mu, \quad \phi = \phi(u) \equiv \log \lambda, \quad a = a(u) \]

- UV: Asymptotically AdS$_5$ as \( u \to -\infty \): \( A(u) \sim -u/\ell \)

\[ \phi \sim \begin{cases} 
\phi_- e^{\Delta_- u} & \Delta = d - \Delta_- \\
- \log(-b_0 u) & \text{IHQCD}
\end{cases} \quad a(r) \to a_{uv} + Q e^{4u} \]
Axionic domain-wall solutions

\[ ds^2 = du^2 + e^{2A(u)} dx^\mu dx_\mu, \quad \phi = \phi(u) \equiv \log \lambda, \quad a = a(u) \]

- UV: Asymptotically $AdS_5$ as $u \to -\infty$: $A(u) \sim -u/\ell$

$\phi \sim \begin{cases} 
\phi e^{\Delta - u} & \Delta = d - \Delta_- \\
- \log(-b_0 u) & \text{IHQCD}
\end{cases} \quad a(r) \rightarrow a_{uv} + Q e^{4u}$

- source term: $a_{uv} = \theta/N$
- vev term: $Q \propto \langle q \rangle$
Axionic domain-wall solutions

\[ ds^2 = du^2 + e^{2A(u)} dx^\mu dx_\mu, \quad \phi = \phi(u) \equiv \log \lambda, \quad a = a(u) \]

- UV: Asymptotically \( AdS_5 \) as \( u \to -\infty \): \( A(u) \sim -u/\ell \)

\[ \phi \sim \begin{cases} \phi_- e^{\Delta_- u} & \Delta = d - \Delta_- \\ - \log(-b_0 u) & \text{IHQCD} \end{cases} \quad a(r) \to a_{uv} + Q e^{4u} \]

- IR: \( e^A \to 0 \). IR Axion regularity requirement

\[ a(u_{ir}) = 0 \]

- Analogy to top-down models where \( a \) comes from a form;
- Required by holographic consistency.
Axion RG flow

\[ \partial_u (Y e^{dA} \dot{a}) = 0 \]

\[ \dot{a} = \frac{Q}{Ye^{3A}} \quad \rightarrow \quad a = a_{uv} + Q \int_{u_{uv}}^{u} \frac{du}{Ye^{3A}} \]
Axion RG flow

$$\partial_u (Y e^{dA} \dot{a}) = 0$$

$$\dot{a} = \frac{Q}{Ye^{3A}} \to a = a_{uv} + Q \int_{u_{uv}}^u \frac{du}{Ye^{3A}}$$

$$a(u_{ir}) = 0 \to a_{uv} = -Q \int_{u_{uv}}^{u_{ir}} \frac{du}{Ye^{3A}}$$
Axion RG flow

\[ \partial_u (Y e^{dA} \dot{a}) = 0 \]

\[ \dot{a} = \frac{Q}{Ye^{3A}} \quad \rightarrow \quad a = a_{uv} + Q \int_{u_{uv}}^{u} \frac{du}{Ye^{3A}} \]

\[ a(u_{ir}) = 0 \quad \Rightarrow \quad a_{uv} = -Q \int_{u_{uv}}^{u_{ir}} \frac{du}{Ye^{3A}} \]

Substitute \( \dot{a} \) in the field equations \( \Rightarrow \) all is controlled by \( Q \)

- \( Y(\phi) \) must diverge faster than \( e^{3A} \rightarrow 0 \) in the IR. Otherwise only trivial axion solution with \( Q = 0 \) makes sense

- Theories with IR AdS fixed point at finite \( \phi = \phi_{ir} \) generically only have trivial axion solutions, \( a = 0 \)

From now on assume IR is at \( \phi \rightarrow \infty \) (good singularity).
Periodicity

The boundary field theory $\theta$ parameter is periodic but the bulk axion may not be

$$a_{uv} = \frac{\theta + 2k\pi}{N}$$

A given value $\theta \in [0, 2\pi]$ corresponds to many values for the axion source term $a_{uv}$

$$\downarrow$$

Multiple (infinite?) holographic RG-flow solutions for a given $\theta$
Backreaction

Take model such that in the IR:

\[ \phi \to +\infty, \quad V(\phi) \simeq -V_{\infty} e^{b\phi}, \quad Y(\phi) \simeq Y_{\infty} e^{\gamma \phi} \]

Two kinds of solutions:

1. \( a(u) \) gives subleading contribution in the IR \( \Rightarrow \) IR axion regularity only fixes \( Q \) in terms of \( a_{uv} \).

2. \( a(u) \) backreacts at leading order and the value of \( Q \) is completely fixed
Backreaction

Take model such that in the IR:

\[ \phi \to +\infty, \quad V(\phi) \simeq -V_\infty e^{b\phi}, \quad Y(\phi) \simeq Y_\infty e^{\gamma\phi} \]

Two kinds of solutions:

1. \( a(u) \) gives subleading contribution in the IR \( \Rightarrow \) IR axion
   regularity only fixes \( Q \) in terms of \( a_{uv} \).

2. \( a(u) \) backreacts at leading order and the value of \( Q \) is
   completely fixed
   - Type-2 solutions not acceptable (we cannot fix the source to
     arbitrary value)
   - For type-1 solutions the backreaction is only important in the
     interior, but becomes negligible in both the UV and the IR.
Free energy

Free energy $\mathcal{F} = \text{Euclidean on-shell action } S_E$

$$S^{(\text{ren})}_{\text{on-shell}}[m, a_{uv}] = - M_p^3 V_d \left( e^{4A} \dot{A} - S_{ct} \right)_{uv} = - V_d (M_p \ell)^3 m^4 C(a_{uv})$$

$$m = (\phi_-)^{1/\Delta} \text{ or } \Lambda_{QCD} \quad C(a_{uv}) \propto \frac{1}{(M_p \ell)^3} \frac{\langle O_\phi \rangle}{m^\Delta}$$
Free energy

Free energy $\mathcal{F} =$ Euclidean on-shell action $S_E$

$$S^{(\text{ren})}_{\text{on-shell}}[m, a_{uv}] = -M_p^3 V_d \left( e^{4A} \dot{A} - S_{ct} \right)_{uv} = -V_d (M_p \ell)^3 m^4 C(a_{uv})$$

$$m = (\phi^-)^{1/\Delta} \quad \text{or} \quad \Lambda_{QCD} \quad \quad C(a_{uv}) \propto \frac{1}{(M_p \ell)^3} \frac{\langle O_\phi \rangle}{m^\Delta}$$

Multiple saddle points: \quad \quad a_{uv}^{(k)} = (\theta + 2\pi k)/N

$$\mathcal{F}_k = -V_d (M_p \ell)^3 m^4 C \left( \frac{\theta + 2\pi k}{N} \right) \quad \quad \mathcal{F}(\theta) = \min_k \mathcal{F}_k$$
Free energy

Free energy $\mathcal{F} = \text{Euclidean on-shell action } S_E$

$$S^{(\text{ren})}_{\text{on-shell}}[m, a_{uv}] = -M_p^3 V_d \left( e^{4A} \dot{A} - S_{ct} \right)_{uv} = -V_d (M_p \ell)^3 m^4 C(a_{uv})$$

$$m = (\phi_-)^{1/\Delta} \text{ or } \Lambda_{\text{QCD}} \quad C(a_{uv}) \propto \frac{1}{(M_p \ell)^3} \frac{\langle O_\phi \rangle}{m^\Delta}$$

Multiple saddle points: \[ a_{uv}^{(k)} = (\theta + 2\pi k)/N \]

$$\mathcal{F}_k = -V_d (M_p \ell)^3 m^4 C \left( \frac{\theta + 2\pi k}{N} \right) \quad \mathcal{F}(\theta) = \text{Min}_k \mathcal{F}_k$$

Expand $C$ to quadratic order (small axion)

$$\mathcal{F}(\theta) = \mathcal{F}(0) + \frac{V_d}{2} \chi \text{Min}_k (\theta + 2\pi k)^2 \quad \chi = \frac{(M_p \ell)^3}{N^2} \left( \int_{uv}^{ir} \frac{du}{Ze^{4A}} \right)^{-1}$$
Free energy

Free energy $\mathcal{F} = \text{Euclidean on-shell action } S_E$

$$S_{\text{on-shell}}^{(\text{ren})}[m, a_{uv}] = -M_p^3 V_d \left( e^{4A} \dot{A} - S_{ct} \right)_{uv} = -V_d (M_p \ell)^3 m^4 C(a_{uv})$$

$$m = (\phi_-)^{1/\Delta_-} \text{ or } \Lambda_{QCD} \quad C(a_{uv}) \propto \frac{1}{(M_p \ell)^3} \frac{\langle O_\phi \rangle}{m^\Delta}$$

Multiple saddle points: $a_{uv}^{(k)} = (\theta + 2\pi k)/N$

$$\mathcal{F}_k = -V_d (M_p \ell)^3 m^4 C \left( \frac{\theta + 2\pi k}{N} \right) \quad \mathcal{F}(\theta) = \text{Min}_k \mathcal{F}_k$$

Expand $C$ to quadratic order (small axion)

$$\mathcal{F}(\theta) = \mathcal{F}(0) + \frac{V_d}{2} \chi \text{Min}_k (\theta + 2\pi k)^2$$

$$\chi = \frac{(M_p \ell)^3}{N^2} \left( \int_{uv}^{ir} \frac{du}{Ze^{4A}} \right)^{-1}$$
Example

Model with a UV fixed point at $\phi = 0$ deformed by relevant operator

$$V = -\frac{1}{\ell^2} \left[ d(d-1) + \left( \frac{1}{2} (d - \Delta_-) \Delta_- - b^2 V_\infty \right) \phi^2 + 4V_\infty \sinh^2 \left( \frac{b\phi}{2} \right) \right], \quad Y = e^{\gamma\phi},$$
Example

Model with a UV fixed point at $\phi = 0$ deformed by relevant operator

$$V = -\frac{1}{\ell^2} \left[ d(d-1) + \left( \frac{1}{2} (d-\Delta) \Delta - b^2 V_{\infty} \right) \phi^2 + 4V_{\infty} \sinh^2 \left( \frac{b\phi}{2} \right) \right], \quad Y = e^{\gamma \phi},$$

$$(W \equiv \dot{A}; \; D = \text{an IR parameter related to } Q)$$
Finite axion range

In the probe approximation, any value of $a_{uv}$ is possible $\Rightarrow$ An infinite number of vacua for every $\theta$
Finite axion range

In the probe approximation, any value of $a_{uv}$ is possible $\Rightarrow$ An infinite number of vacua for every $\theta$

$$a_0^{uv}, \ b=\frac{13}{10}, \ y=1$$

Backreaction $\Rightarrow$ range of $a$ becomes bounded
Finite axion range

Range of $a$ bounded $\Rightarrow$ finite number of vacua.

$$|a_{uv}| \leq \int_{uv}^{ir} \frac{1}{\sqrt{Y(\phi)}} \sim O(1)$$

$\Rightarrow$ There are $O(N)$ vacua.
PART II

Role of contact terms in holographic correlators

- Connect the full topological charge correlator $\langle q(x)q(0) \rangle$ to the spectral data for axial glueballs (masses and decay constants)
- Compute reliably the position-space correlator: exercice in distributional Fourier transform
- Crucial: understand the role of contact terms

Work sparked by discussion with E. Kiritsis, U. Gursoy, I. Iatrakis A. Schaefer, S-W. Mages and others to match the topological correlator with lattice
Topological correlator

\[ S_{QFT} = S_0 + N \int d^4x \, \alpha(x) q(x), \quad \alpha \equiv \frac{\theta}{N} \]

Wick rotation to Euclidean: \( q(x) \rightarrow iq(x) \)

\[ Z[\alpha] = \int [dA] \, e^{-S_E + iN \int \alpha(x) q(x)} \]
Topological correlator

\[ S_{QFT} = S_0 + N \int d^4x \, \alpha(x) q(x), \quad \alpha \equiv \frac{\theta}{N} \]

Wick rotation to Euclidean: \( q(x) \rightarrow iq(x) \)

\[ Z[\alpha] = \int [dA] \, e^{-S_E + iN \int \alpha(x) q(x)} \]

Expand \( Z \) to quadratic order in \( \alpha \):

\[ Z[\alpha] \simeq \exp \left[ -\frac{N^2}{2} \int \alpha(x) G(x, y) \alpha(y) \right] \quad G(x, y) = \langle q(x) q(y) \rangle \]

Topological susceptibility: take \( \alpha = \theta/N = \text{const} \)

\[ \mathcal{F}(\theta) - \mathcal{F}(0) = -\frac{1}{V} \log \frac{Z[\theta/N]}{Z[0]} = \frac{1}{2} \chi \theta^2 \quad \chi = \int d^4x \, G(x) \]
Topological correlator from holography

\[ a(x, r) \Leftrightarrow Nq(x) \equiv \frac{N}{16\pi^2} Tr F \tilde{F}(x) \]

\[ a(x, r) \simeq_{r \to 0} \alpha(x) + \ldots \Leftrightarrow S_{QFT} = S_0 + N \int d^4x \alpha(x)q(x), \quad \alpha \equiv \frac{\theta}{N} \]
Topological correlator from holography

\[ a(x, r) \Leftrightarrow Nq(x) \equiv \frac{N}{16\pi^2} \text{Tr} F\tilde{F}(x) \]

\[ a(x, r) \simeq_{r \to 0} \alpha(x) + \ldots \Leftrightarrow S_{QFT} = S_0 + N \int d^4x \alpha(x)q(x), \quad \alpha \equiv \frac{\theta}{N} \]

Wick rotation to Euclidean: \( q(x) \rightarrow iq(x) \)

\[ Z[\alpha] = \int [dA] e^{-S_E + iN \int \alpha(x)q(x)} = \exp i\left\{ -S_E^{\text{grav}}[\alpha(x; r)] \right\}_{a(x,r) \rightarrow \alpha(x)} \]
Topological correlator from holography

\[ a(x, r) \Leftrightarrow Nq(x) \equiv \frac{N}{16\pi^2} \text{Tr} F\tilde{F}(x) \]

\[ a(x, r) \sim_{r\to 0} \alpha(x) + \ldots \Leftrightarrow S_{QFT} = S_0 + N \int d^4x \alpha(x)q(x), \quad \alpha \equiv \frac{\theta}{N} \]

Wick rotation to Euclidean: \( q(x) \to iq(x) \)

\[ Z[\alpha] = \int [dA] e^{-S_E+iN \int \alpha(x)q(x)} = \exp i\left\{-S_E^{\text{grav}}[\alpha(x); r]\right\}_{a(x, r) \to \alpha(x)} \]

- Euclidean dictionary: \( a(x, r) \) dual to \( iNq(x) \).
- Reflection positivity for a pseudoscalar operator:

\[ G(x) \equiv \langle q(x)q(0) \rangle < 0 \quad x \neq 0 \]
Topological correlator from holography

\[ S_E[a] = M_p^3 \int d^4x \, dr \, \frac{Z(r)}{2} \left[ (\partial_r a)^2 + \partial_{\mu} a \partial^{\mu} a \right], \]

\[ Z(r) = e^{3A(r)} Y(\phi(r)) \quad (M_p \ell)^3 \sim N^2 \]
Topological correlator from holography

\[ S_E[a] = M_p^3 \int d^4x \, dr \, \frac{Z(r)}{2} \left[ (\partial_r a)^2 + \partial_\mu a \partial^\mu a \right], \]

\[ Z(r) = e^{3A(r)} Y(\phi(r)) \quad (M_p \ell)^3 \sim N^2 \]

Axion fluctuations are probes on a background \((g_{\mu\nu}, \phi)\),

\[ g_{\mu\nu} = e^A(r) \left[ dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu \right], \quad \lambda = \lambda(r) \]

\[ \partial_r [Z(r) \partial_r a(x, r)] + Z(r) \partial_\mu \partial^\mu a(x, r) = 0 \]

\[ S_E[a] = -M_p^3 \lim_{r \to 0} \left[ \int d^4x \, \frac{Z(r)}{2} a(x, r) \partial_r a(x, r) \right] > 0 \]

\[ = \frac{N^2}{2} \int d^4x d^4y \, \alpha(x) G(x, y) \alpha(y) \]
Two-point function from spectrum

Write two-point correlator in terms of exchange of axial glueballs

\[ \langle q(x)q(y) \rangle = \sum \tilde{G}(k) = -\sum_{n=0}^{\infty} \frac{f_n^2}{(k^2 + m_n^2)} \]

Spectral data \((f_n, m_n)\) easily accessible from holography.
Two-point function from spectrum

Write two-point correlator in terms of exchange of axial glueballs

\[ \langle q(x)q(y) \rangle = \sum \tilde{G}(k) = -\sum_{n=0}^{\infty} \frac{f_n^2}{k^2 + m_n^2} \]

Spectral data \((f_n, m_n)\) easily accessible from holography.

PROBLEM:

- \(G(x) < 0\) but \(\chi_{\text{top}} = \int d^4 x G(x) = \tilde{G}(k = 0)\) positive.
Two-point function from spectrum

Write two-point correlator in terms of exchange of axial glueballs

\[ \langle q(x)q(y) \rangle = \sum \tilde{G}(k) = C_0 - \sum_{n=0}^{\infty} \left( \frac{f_n^2}{k^2 + m_n^2} \right) \]

Spectral data \((f_n, m_n)\) easily accessible from holography.

**PROBLEM:**

- \(G(x) < 0\) but \(\chi_{top} = \int d^4x G(x) = \tilde{G}(k = 0)\) positive.
  - Well known solution: there must be a contact term which gives a positive contribution at \(k = 0\).
Two-point function from spectrum

Write two-point correlator in terms of exchange of axial glueballs

\[ \langle q(x)q(y) \rangle = \sum \mathbf{X} \]

\[ \tilde{G}(k) = C_0 - \sum_{n=0}^{\infty} \frac{f_n^2}{(k^2 + m_n^2)} \]

Spectral data \((f_n, m_n)\) easily accessible from holography.

PROBLEM:

- \(G(x) < 0\) but \(\chi_{top} = \int d^4 x G(x) = \tilde{G}(k = 0)\) positive.
  Well known solution: there must be a contact term which gives a positive contribution at \(k = 0\).
Two-point function from spectrum

Two point correlator at finite separation can be understood in terms of exchange of axial glueballs

\[ G(x) = C_0 \delta^{(4)}(x) - \Gamma[x] \]
Two-point function from spectrum

\[ \tilde{G}(k) = C_0 - \sum_{n=0}^{\infty} \frac{f_n^2}{(k^2 + m_n^2)} \]

Spectral data \((f_n, m_n)\) easily accessible from holography.
Two-point function from spectrum

\[ \tilde{G}(k) = C_0 - \sum_{n=0}^{\infty} \frac{f_n^2}{(k^2 + m_n^2)} \]

Spectral data \((f_n, m_n)\) easily accessible from holography.

PROBLEM:

- At any finite \(k\), the series diverges: on general grounds since \(G(k) \sim k^4 \log k^2\) in the UV: need \(m_n^2 \sim f_n \sim n\).

How do we relate the (finite) holographic correlator to the spectrum?
Axial Glueball Spectrum

- Go to momentum space (and back to Lorentzian signature):

\[ a(x, r) = a_k(r)e^{ik\mu x^\mu} \Rightarrow a''_k + \frac{Z'}{Z}a'_k - k^2 a_k = 0, \quad k^2 = -E^2 + |\vec{k}|^2 \]

- Define: \( \psi(r) = \sqrt{Z(r)}a_k(r) \).

\[
\psi'' - \left[ \frac{1}{2} \frac{Z''}{Z} - \frac{1}{4} \left( \frac{Z'}{Z} \right)^2 \right] \psi - k^2 \psi = 0
\]

Effective Schrödinger equation:

\[-\psi''(r) + V(r)\psi = m^2\psi, \quad k^2 = -m^2, \quad V(r) = Z^{-1/2} \left( Z^{1/2} \right)''\]
Axial Glueball Spectrum

\[ V(r) \sim \begin{cases} \frac{1}{r^2} & r \to 0 \\ r^2 & r \to +\infty \end{cases} \]

- discrete tower of normalizable modes with masses \( m_n \)
- The residues (glueball decay constants) \( f_n \) are given by:

\[ f_n = \sqrt{Z(0)} \left| \psi_n'(0) - \frac{1}{2} \frac{Z'(0)}{Z(0)} \psi_n(0) \right| \]

\[ m_n^2 \sim n, \quad f_n \sim n \quad \Rightarrow \quad \sum \frac{f_n^2}{(k^2 + m_n^2)} \sim \sum_{n > k^2} n = \infty \]

- The sum of terms with \( n < k^2 \) gives at large \( n \) a result \( \sim k^4 \) which is roughly correct.
The importance of contact terms

\[
\tilde{G}(k) = \lim_{r \to 0} \left[ Z^{1/2} \partial_r \left( Z^{-1/2} \Psi_k \right) \right], \quad \lim_{r \to 0} \frac{\Psi_k(r)}{Z^{1/2}(r)} = 1.
\]

\(\Psi_k\) non-normalizable \(\Rightarrow\) cannot expand \(\Psi_k\) in sum of normalizable eigenmodes.

**FIX:** extract enough powers of \(k^2\) from \(\Psi_k\) so that what’s left is normalizable:

\[
\Psi_k(r) = \psi_0(r) + k^2 \psi_2(r) + k^4 \psi_4(r) + \sum_n c_n(k) \psi_n(r)
\]

\[
\tilde{G}(k) = C_0 + C_2 k^2 + C_4 k^4 - k^6 \sum_n \frac{f_n}{m_n^6 (k^2 + m_n^2)}
\]

\[
G(x) = C_0 \delta(x) + C_2 \Box \delta(x) + C_4 \Box^2 \delta(x) - \Gamma(x)
\]
Contact terms from holography

\[ G(k) = C_0 + C_2 k^2 + C_4 k^4 + k^6 \sum_n \frac{f_n}{m_n^6 (k^2 + m_n^2)} \]

Can compute iteratively from axion equation in small-\( k \) expansion:

\[ \partial_r [Z a_k(r)] = k^2 a_k(r) \quad a_k(0) = 1, \quad a_k(+\infty) = 0 \]

\[ a_k(r) = a_0(r) + k^2 a_1(r) + k^4 a_2(r) + \ldots \]

\[ C_0 \equiv \chi_{top} = \left( \int_0^{r_{ir}} \frac{1}{Z} \right)^{-1}, \quad a_0(r) = 1 - \chi_{top} \int_0^r \frac{1}{Z} \]

\[ C_{2n} = Z(0)a'_n(0) \quad a_n(r) = \int_0^r \frac{dr'}{Z(r')} \int_{r_{ir}}^{r'} dr'' Z(r'') a_{n-1}(r'') \]
Renormalization and ambiguities

- $C_0$ is finite and unambiguous;
- $C_2$ and $C_4$ have quadratic and log divergences.
- Holographic renormalization removes them up to scheme-dependent finite counterterms

\[ S^{(2)}_{ct} = C_2^{ct} \int_{r=\epsilon} d^4x \sqrt{\gamma} \phi^{\frac{d-2}{\Delta}} \gamma^{\mu\nu} \partial_\mu a \partial_\nu a \rightarrow C_2^{ct} \int d^4x \, m^2 \partial_\mu a \partial^\mu a \]

\[ S^{(4)}_{ct} = C_4^{ct} \int_{r=\epsilon} d^4x \sqrt{\gamma} \gamma^{\mu\nu} \gamma^{\rho\sigma} (\partial_\mu \partial_\nu a)(\partial_\rho \partial_\sigma a) \rightarrow C_4^{ct} \int d^4x \, (\partial_\mu \partial^\mu a)^2 \]

To match the correlator with lattice we must fix this scheme-dependence.
Conclusion

Axion dynamics in AdS/CFT:

- Rich phenomenology in axial sector
  - Running of $\theta$-angle
  - Multiple vacua
- Clean environment to compare with Lattice
  - Axial glueball spectrum
  - Two-point correlator
  - Precise identification of contact terms
Towards the real world

Take a background generated by a single scalar $\lambda$, dual to $Tr F^2$, and representing the running t’Hooft coupling Gursoy, Kiritsis, Mazzanti, FN 08-09

$$S_{bkg} = N^2 \int d^5 x \sqrt{-g} \left[ R - \frac{4}{3} \frac{(\partial \lambda)^2}{\lambda^2} + V(\lambda) \right]$$
Towards the real world

Take a background generated by a single scalar $\lambda$, dual to $Tr F^2$, and representing the running t’Hooft coupling Gursoy, Kiritsis, Mazzanti, FN 08-09

$$S_{bkg} = N^2 \int d^5 x \sqrt{-g} \left[ R - \frac{4}{3} \frac{(\partial \lambda)^2}{\lambda^2} + V(\lambda) \right]$$

- UV asymptotic freedom, confinement, $0^{++}$ and $2^{++}$ glueball spectrum and thermodynamics in agreement with lattice, can be achieved by an appropriate choice of $V(\lambda)$.
Towards the real world

Take a background generated by a single scalar $\lambda$, dual to $Tr F^2$, and representing the running t’Hooft coupling Gursoy, Kiritsis, Mazzanti, FN 08-09

$$S_{bkg} = N^2 \int d^5 x \sqrt{-g} \left[ R - \frac{4}{3} \frac{(\partial \lambda)^2}{\lambda^2} + V(\lambda) \right]$$

- UV asymptotic freedom, confinement, $0^{++}$ and $2^{++}$ glueball spectrum and thermodynamics in agreement with lattice, can be achieved by an appropriate choice of $V(\lambda)$.
- The solution has asymptotics:

$$e^{A(r)} \sim \begin{cases} \frac{\ell^2}{\rho^2} & r \to 0 \\ e^{-2\Lambda^2 r^2} & r \to \infty \end{cases} \quad \lambda(r) \sim \begin{cases} \frac{1}{\beta_0 \ln r} & r \to 0 \\ r e^{3\Lambda^2 r^2/2} & r \to \infty \end{cases}$$
Parametrizing the axion Lagrangian

\[ S_\alpha = \frac{1}{2} \int \sqrt{-g} Y(\lambda) (\partial \alpha)^2 \]

\[ Y(\lambda) = Y_0 \left( 1 + c_1 \lambda + c_4 \lambda^4 \right) \]
Parametrizing the axion Lagrangian

\[ S_a = \frac{1}{2} \int \sqrt{-g} Y(\lambda) (\partial a)^2 \]

\[ Y(\lambda) = Y_0 \left( 1 + c_1 \lambda + c_4 \lambda^4 \right) \]

↓ ↓

Finite \( \chi_{top} \) Universal Regge slopes
Parametrizing the axion Lagrangian

\[ S_a = \frac{1}{2} \int \sqrt{-g} Y(\lambda)(\partial a)^2 \]

\[ Y(\lambda) = Y_0 \left( 1 + c_1 \lambda + c_4 \lambda^4 \right) \]

↓ ↓ ↓

Free parameters to fix by matching lattice/experiment
Parametrizing the axion Lagrangian

\[ S_a = \frac{1}{2} \int \sqrt{-g} Y(\lambda) (\partial \alpha)^2 \]

\[ Y(\lambda) = Y_0 \left( 1 + c_1 \lambda + c_4 \lambda^4 \right) \]

Discrete $0^{--}$ spectrum with asymptotics (from WKB method)

\[ m_n^2 \sim n, \quad f_n \sim n \]
Parametrizing the axion Lagrangian

\[ S_a = \frac{1}{2} \int \sqrt{-g} Y(\lambda) (\partial a)^2 \]

\[ Y(\lambda) = Y_0 \left( 1 + c_1 \lambda + c_4 \lambda^4 \right) \]

Discrete $0^{-+}$ spectrum with asymptotics (from WKB method)

\[ m_n^2 \sim n, \quad f_n \sim n \]

For $c_1 = 0, c_4 = 0.26$ one finds a good match with Lattice result for the lowest lying $0^{-+}$ states.

<table>
<thead>
<tr>
<th></th>
<th>5d model</th>
<th>lattice hep-lat/9901004</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{0^{-+}}/m_{0^{++}}$</td>
<td>1.50</td>
<td>1.50(4)</td>
</tr>
<tr>
<td>$m_{0^{*-+}}/m_{0^{++}}$</td>
<td>2.10</td>
<td>2.11(6)</td>
</tr>
</tbody>
</table>
Matching the lattice $0^{-+}$ spectrum

The spectrum is rather insensitive to the details of $Y(\lambda)$.

Changing $c_1$ between zero and 100 (with $c_4$ conditioned to keep $m_{0^{-+}}$ fixed) only affects the first excited state in the tower.
**Full Correlator**

The best shot at testing the model is to compute the full correlator. We can do it in position space, and compare directly with a lattice computation (in progress)

\[
\langle \tilde{O}(x)\tilde{O}(0) \rangle = \square^3 \left( \frac{1}{|x|} \sum_{n=0}^{\infty} \frac{f_n^2}{m_n^5} K_1(m_n |x|) \right)
\]

the plots correspond to the two point function with \( c_1 = 0 \) and \( c_1 = 5 \).