Fixed point collisions and tensorial order parameters in Luttinger semimetals and some popular field theories

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Outline:

1) Condensed matter motivation: (symmetry-poor real world)
   Quadratic band touching in three dimensions, and the Luttinger Hamiltonian
   Coulomb interaction and the scale-invariant ("non-Fermi liquid") fixed point
   Fixed point annihilation and the concomitant separation of scales
   Spin-2 (tensor) order parameter

2) Field theory applications
   Chiral symmetry breaking in $\text{QED}_{d<4}$ revisited
   Interacting $O(N)$ field theory above four dimensions
Gapless semiconductors with band inversion (gray tin, HgTe, YPtBi)

Luttinger (spin-orbit) Hamiltonian ($p$-orbitals, $J = 3/2$) \cite{Luttinger1956}

\[
H = \frac{1}{2m} \left( \left( \gamma_1 + \frac{5}{2} \gamma_2 \right) k^2 - 2 \gamma_2 (\mathbf{k} \cdot \mathbf{S})^2 \right)
\]

with (rotationally symmetric) eigenvalues

\[
E_L(k) = \frac{\gamma_1 + 2\gamma_2}{2m} k^2, \quad E_H(k) = \frac{\gamma_1 - 2\gamma_2}{2m} k^2
\]
Luttinger Hamiltonian \( \text{a la} \) Dirac:

\[
H(k) = \epsilon(k) + \frac{\gamma^2}{m} d_a \Gamma^a
\]

where,

\[
\epsilon(k) = \frac{\gamma^1}{2m} k^2, \quad d_a(k) = -3 \xi^{ij}_{a} k_i k_j,
\]

\[
d_1 = -\sqrt{3} k_y k_z, \quad d_2 = -\sqrt{3} k_x k_z, \quad d_3 = -\sqrt{3} k_x k_y
\]

\[
d_4 = -\frac{\sqrt{3}}{2} (k_x^2 - k_y^2),
\]

\[
d_5 = -\frac{1}{2} (2 k_z^2 - k_x^2 - k_y^2).
\]

are \( l=2 \) (real) spherical harmonics.

**Five 4 \times 4** Dirac matrices satisfy Clifford algebra:

\[
\{\Gamma^a, \Gamma^b\} = 2\delta_{ab}
\]
Long-range Coulomb interaction \((1/q^2)\):

without the hole band, at ``zero'' (low) density:

Wigner crystal

With the hole band filled and particle band empty: the system is critical!

In the RG language, the charge flows with the change of cutoff, to one loop:

\[
\frac{de^2}{d \ln b} = (z + 2 - d)e^2 - 4e^4
\]

(Abrimosov, ZETF 1974; Moon, Xu, Kim, Balents, PRL 2013)
Below and near the upper critical dimension, $d_{up} = 4$, the flow is towards a non-Fermi liquid fixed point, with the charge at

$$e^2_* = 15\epsilon/76 + \mathcal{O}(\epsilon^2)$$

$$\epsilon = 4 - d$$

and the dynamical critical exponent $z < 2$:

$$z = 2 - \frac{16}{15}e^2$$

This implies power-laws in various responses, such as specific heat:

$$c_v \sim T^{d/z} \approx T^{1.7}$$

Emergent scale (conformal?) invariance!

Cheap way to get a non-Fermi liquid phase in 3D!?

\[ L = L_0 + L_a + L_\psi \]

with the free (Luttinger) part,

\[ L_0 = \psi_i^\dagger [\partial_r + H_0(-i\nabla)] \psi_i \]

and long-range (Coulomb) and short-range (Coulomb) interactions

\[ L_a = \frac{1}{2}(\nabla a)^2 + iea \psi_i^\dagger \psi_i \]

\[ L_\psi = g_1 (\psi_i^\dagger \psi_i)^2 + g_2 (\psi_i^\dagger \gamma a \psi_i)^2 + g_3 (\psi_i^\dagger \gamma_{ab} \psi_i)^2 \]
The RG flow of all the couplings: (one loop)

\[
\frac{de^2}{d \ln b} = (2 + z - d - \eta_a) e^2, \\
\frac{dg_1}{d \ln b} = (z - d) g_1 - (e^2 + 2g_1) g_2 - 24g_3^2, \\
\frac{dg_2}{d \ln b} = (z - d) g_2 + \frac{4(e^2 + 2g_1)g_2}{5} - \frac{(e^2 + 2g_1)^2}{20} \\
- \frac{37 + 16N}{5} g_2^2 + \frac{112}{5} g_2 g_3 - \frac{136}{5} g_3^2, \\
\frac{dg_3}{d \ln b} = (z - d) g_3 - \frac{1}{5} (e^2 + 2g_1) g_3 + g_2^2 - 6g_2 g_3 \\
+ \frac{4(11 - 4N)}{5} g_3^2
\]

with,

\[
\eta_a = N e^2 \quad z = 2 - \frac{4}{15} e^2
\]

and the “charge”

\[
e^2 = 2m e_{el}^2 / (4\pi\hbar^2 \varepsilon)
\]
One-loop at face value:
close to and below $d=4$ there is a (IR stable) NFL fixed point, but also
a (mixed-stable) quantum critical point at strong interaction:

They get closer, but remain separated in the coupling space!
At some “lower” (non-integer) critical dimension $d = d_1$ NFL and QCP collide:

In one-loop calculation this occurs at $d_l = 3.26240$, slightly above three dimensions.
Finally, below the NFL and QCP become complex, and there is only a runaway flow left in the physical (real) coupling space:

\( g_2 \) diverges at a finite scale: the system is unstable, towards a new phase (gapped (Mott) insulator).

Scale invariance lost!
Fixed point collision and annihilation:

General number of fermions ($N$) and dimension ($d$):

Near $d=2$ the collision occurs in the completely perturbative regime:

$$N \geq N_c(\epsilon) = \frac{64}{25\epsilon^2} + \mathcal{O}(1/\epsilon)$$

$$\epsilon = d - 2$$
Critical number of fermions in $d=3$:

TABLE III. Critical fermion number $N_c$ in $d = 3$ spatial dimensions from different approaches.

<table>
<thead>
<tr>
<th>Method</th>
<th>Reference</th>
<th>$N_c(d = 3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2 + \epsilon$ expansion</td>
<td>Sec. III</td>
<td>2.56</td>
</tr>
<tr>
<td>RG in fixed $d = 3$</td>
<td>Sec. IV</td>
<td>2.10</td>
</tr>
<tr>
<td>Functional RG</td>
<td>Sec. VI</td>
<td>1.86</td>
</tr>
<tr>
<td>$1/N$ expansion in $d = 3$</td>
<td>Ref. [18]</td>
<td>$\geq 2.6(2)$</td>
</tr>
</tbody>
</table>

(Janssen & IH, PRB 2017)
Order parameter for $d < d_{\text{low}}$

\[ \chi_i = 2g_2 \langle \Psi^\dagger \gamma_i \Psi \rangle \]

Out of the five $\chi_1, \ldots, \chi_5$ not all equivalent:

1. $\chi_1 \neq 0$: $\varepsilon(\bar{p})$ gapped with minimal gap at two opposite points on equator

2. $\chi_5 < 0$: $\varepsilon(\bar{p})$ gapless with gap closing at north and south pole

3. $\chi_5 > 0$: $\varepsilon(\bar{p})$ gapped with minimal gap at entire equator

Energy $E = \int \frac{d\bar{p}}{(2\pi)^3} \varepsilon(\bar{p})$ is minimized for (3): $\chi_5 > 0$ (modulo $O(3)$)
The fate of NFL: if $d$ is above but close to $d=3$, the flow becomes slow close to (complex!) NFL fixed point. The RG escape time is long:

$$b_0 = e^{\frac{c}{\sqrt{d_{10w}-d}}} - B + O(d_{10w}-d)$$

with non-universal constants $C$ and $B$. There is wide crossover region of the NFL behavior within the temperature window

$$(T_c, T_*)$$

with the critical temperature,

$$T_c \approx T_* b_0^{-z}$$

Characteristic energy scale for interaction effects

$$k_B T_* \sim \frac{e_{el}^2}{\varepsilon L_*} = \frac{\hbar^2}{2mL_*^2} = \frac{4m}{m_{el} \varepsilon^2} E_0$$

(Sherrington & Kohn, Halperin & Rice, RMP 1968)
Some numbers: (for HgTe)

small mass \[ \frac{m}{m_{el}} \approx \frac{1}{50} \]

high dielectric constant \[ \varepsilon \approx 30 \]

still a reasonable \[ T_* \sim 10 \text{ K} - 100 \text{ K} \]

and (maybe) a detectable \[ T_c \approx T_*/100 \]
Cubic symmetry: realistic Luttinger Hamiltonian, cubic symmetry

\[
H = \frac{\hbar^2}{2m^*} \left[ \left( \alpha_1 + \frac{5}{2} \alpha_2 \right) p^2 \mathbf{1}_4 - 2 \alpha_3 (\mathbf{p} \cdot \mathbf{J})^2 \right. \\
+ \left. 2 (\alpha_3 - \alpha_2) \sum_{i=1}^{3} p_i^2 J_i^2 \right],
\]

contains particle-hole asymmetry and anisotropy parameters

\[
x = -\frac{\alpha_1}{\alpha_2 + \alpha_3}, \quad \delta = -\frac{\alpha_2 - \alpha_3}{\alpha_2 + \alpha_3}
\]

with (generically) particularly slow flow of the anisotropy:

\[
\dot{\delta} \simeq -\frac{8}{105} e^2 \delta.
\]
Flow of the charge is now

\[ \dot{e}^2 = \frac{de^2}{d \log b} = (4 - d - \eta)e^2 - \frac{f e^2(\delta)}{1 - \delta^2}e^4 \]

which lowers the critical dimension: (Boettcher & IH, PRB 2017)
Chiral symmetry breaking in QED$_3$ revisited

Schwinger-Dyson, large-N, calculation of the mass gap (Appelquist, Nash, Wijewardhana, PRL 1988; Pisarski, PRD 1984):

$$\Sigma(0) = \alpha e^{(\delta + 2)} \exp \left[ \frac{-2n\pi}{(32/\pi^2N - 1)^{1/2}} \right]$$

as the number of four-component Dirac fermions $N$

$$N \to \frac{32}{\pi^2}$$

from below.

This should also be understandable as a fixed point collision and annihilation.
Consider QED near four space-time dimensions with (generated) quartic terms (Herbut, PRD 2016; Di Pietro et al, PRL 2016)

\[
L = \bar{\Psi}_n i\gamma_\mu (\partial_\mu - ieA_\mu) \Psi_n + \sum_{a=1}^{2} g_a (\bar{\Psi}_n X_a \gamma_\mu \Psi_n)^2 + \frac{F_{\mu\nu}^2}{4}
\]

with

\[
X_1 = 1 \\
X_2 = \gamma_5
\]

i. e. with additional (axial) current – (axial) current interactions.
The flow in the IR ($\Lambda \to \Lambda/b$), one loop: (IH, PRD 2016)

\[
\begin{align*}
\beta_1 &= (2 - d)g_1 + 4(N + 1)g_1^2 - 8g_1g_2 - 6e^2g_2, \\
\beta_2 &= (2 - d)g_2 + 2(2N - 1)g_2^2 + 4g_1g_2 \\
&\quad - 6g_1^2 - 6e^2g_1 - \frac{3}{2}e^4, \\
\beta_e &= (4 - d)e^2 + \beta_{e0}(e).
\end{align*}
\]

and the charge beta-function precisely in $d=4$ is:

\[
\beta_{e0}(e) = -\frac{4N}{3}e^4 - 4Ne^6 + O(Ne^8, N^2e^8)
\]

(Gorishny, Kataev, Larin 1991 (four loop))
Introducing linear combinations:

\[ g_{\pm} = g_1 \pm g_2 \]

Equations (almost) decouple

\[
\beta_+ = (2 - d)g_+ + 2(N - 1)g_+^2 + 2Ng_- - 6g_+e^2 - \frac{3}{2}e^4,
\]

\[
\beta_- = (2 - d)g_- + 6g_-^2 + 4(N + 1)g_+g_- + 6g_-e^2 + \frac{3}{2}e^4.
\]

When \( N=0 \) the first equation decouples. At zero charge:

1) Gaussian stable FP \( g_{\pm} = 0 \)

2) Critical FP \( g_+ = 0, \quad g_- = 1/3 \)

and two more (unimportant) FPs.
Note that when \( g_+ = 0 \), then:

\[
\sum_{a=1}^{2} g_a (\bar{\psi} X_a \gamma_\mu \psi)^2 = -g_- [(\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \psi)^2] 
\]

So a large positive \( g_- \) indeed favors CSB.

Turning on a small charge by hand FP 1 (conformal phase) and FP 2 (critical point for CSB) approach each other.

At one loop and near \( d=4 \) the fixed points collide at

\[
e_c^2 = 3 - 2\sqrt{2} = 0.17157  
\]

At which

\[
g_+ = -e_c^4 / 2 = -0.0147 \quad g_- = e_c^2 / 2 = 0.0857 
\]

at least are reasonably small.
Equating the critical and the IR fixed point value of the charge yields

\[ \frac{4 - d}{N_c} = -\lim_{N \to 0} \frac{\beta_{e_0}(e_c)}{Ne_c^2} \]

and finally

\[ N_c = \frac{3(4 - d)}{4(e_c^2 + 3e_c^4)} \approx 2.88596(4 - d) + O((4 - d)^2) \]

Compares well with other analytical approaches.

Numerically, CSB maybe only at N=0?
(Karthik and Narayan, PRD 2016)
O(N) critical point above four (space-time) dimensions

Above four dimensions Wilson-Fisher fixed point moves to unphysical region and becomes IR unstable (bicritical):

$$\epsilon = 4 - d$$

(IH, A modern approach to critical phenomena (CUP 2007), p. 53)

Can it be understood as an IR stable FP of another theory?
Fei, Giombi, Klebanov (PRD 2014): consider

\[ L = \frac{1}{2} (\partial_\mu z)^2 + \frac{1}{2} (\partial_\mu \phi_i)^2 + g z \phi_i \phi_i + \lambda z^3 \]

which is (log) renormalizable at d=6.

Below d=6 there is an IR stable fixed point for \( d = 6 - \epsilon \)

\[ N_{\text{crit}} = 1038.266 - 609.840 \epsilon - 364.173 \epsilon^2 + \mathcal{O}(\epsilon^3) \]

(Fei, Giombi, Klebanov, Tarnopolsky, PRD 2015)
Alternative formulation (IH and Janssen, PRD 2016)

Consider XY model (\(N=2\)):

\[
(\phi_1^2 + \phi_2^2)^2 = (\phi_1^2 - \phi_2^2)^2 + (2\phi_1\phi_2)^2 \\
= (\phi^T \sigma_3 \phi)^2 + (\phi^T \sigma_1 \phi)^2.
\]

Alternative Hubbard-Stratonovich decoupling

\[
-\frac{g^2}{2}(\phi_1^2 + \phi_2^2)^2 = \frac{1}{2} z_a z_a + g z_a \phi^T \sigma_a \phi \\
a \in \{1, 3\}
\]

to motivate an another representation of the XY model:

\[
L = \frac{1}{2} z_a (m_z^2 - \partial^2_\mu) z_a + \frac{1}{2} \phi_i (m^2_\phi - \partial^2_\mu) \phi_i + g z_a \phi^T \sigma_a \phi
\]
For a general $N$:

$$\frac{1}{2} z_a z_a + g z_a \phi^T \Lambda^a \phi = -\frac{g^2}{2} \phi_i \Lambda^a_{ij} \phi_j \phi_k \Lambda^a_{kl} \phi_l$$

$$a = 1, \ldots, M_N$$

$$M_N = \frac{(N - 1)(N + 2)}{2}$$

is the number of components of second rank irreducible tensor. Completeness of the set of real symmetric $\Lambda^a$ - matrices

$$\Lambda^a_{ij} \Lambda^a_{kl} = \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} - \frac{2}{N} \delta_{ij} \delta_{kl}$$

So that

$$\frac{1}{2} z_a z_a + g z_a \phi^T \Lambda^a \phi = g^2 \left( \frac{1}{N} - 1 \right) (\phi_i \phi_i)^2$$

is just the original quartic term!
Alternative O(N) model: (IH, Janssen, PRD 2016)

\[ L = \frac{1}{2} (\partial_\mu z_a)^2 + \frac{1}{2} (\partial_\mu \phi_i)^2 + g z_a \phi_i \Lambda_{i j}^a \phi_j + \lambda \text{Tr}[(z_a \Lambda^a)^3]. \]

which is also renormalizable in d=6. Right below d=6, one loop:

\[ \frac{d\lambda}{d \ln b} = \frac{1}{2} (\epsilon - 3\eta_z) \lambda + 36 \left( N + 4 - \frac{24}{N} \right) \lambda^3 + \frac{4}{3} g^3, \]

\[ \frac{dg}{d \ln b} = \frac{1}{2} (\epsilon - \eta_z - 2\eta_\phi) g + 4 \left( 1 - \frac{2}{N} \right) g^3 + 12 \left( N + 2 - \frac{8}{N} \right) g^2 \lambda, \]

\[ \eta_z = 12 \left( N + 2 - \frac{8}{N} \right) \lambda^2 + \frac{4}{3} g^2, \quad \eta_\phi = \frac{4}{3} \left( N + 1 - \frac{2}{N} \right) g^2. \]
This flow has an IR stable fixed point for:

\[ 1 < N < 2.6534 \]

and again for

\[ 2.9991 < N < 3.6846 \]

For \( N=2 \):

\[ \eta_\phi = 2\eta_z = \frac{2}{5}e \]

and for \( N=3 \):

\[ \eta_z = \eta_\phi = \frac{5}{33}e \]

and positive!
Flow for $N=3$:

For $3.6847 < N < 4$ the fixed point A becomes stable, but runs to infinity as $N \to 4$. 
At N=3 at the fixed point the theory becomes:

\[ \mathcal{L} = \frac{1}{4} Tr(\partial_\mu M)^2 + \frac{g^2}{6} Tr(M^3) \]

with

\[ M = z_a \Lambda^a + \phi_i S^i \]

and

\[ (S^i)_{jk} = i \epsilon_{ijk} \]

It is invariant under SU(3) transformation:

\[ M \rightarrow U M U^\dagger \]
Four-loop calculation: (Gracey, IH, Roscher, PRD 2018)

\[
\begin{align*}
\beta_1(g_i) &= -\frac{1}{2}\epsilon g_1 - \left[-17g_1^2 + 42g_1g_2 - 7g_2^2\right]\frac{g_1}{36} + \left[-6617g_1^4 + 3591g_1^3g_2 - 1988g_1^2g_2^2 + 2625g_1g_2^3 - 791g_2^4\right]\frac{g_1}{19444} \\
&\quad - \left[4380480\zeta_3 g_1^6 + 2469685g_1^6 - 1959552\zeta_3 g_1^5 g_2 + 9953370g_1^5 g_2 - 9897552\zeta_3 g_1^4 g_2^2 - 3206105g_1^4 g_2^2 \\
&\quad + 10723104\zeta_3 g_1^3 g_2^3 - 3644172g_1^3 g_2^3 + 2830464\zeta_3 g_1^2 g_2^4 + 4790919g_1^2 g_2^4 - 2231712\zeta_3 g_1 g_2^5 \\
&\quad + 4461618g_1 g_2^5 + 371952\zeta_3 g_2^6 - 1217531g_2^6\right]\frac{g_1}{419904}
\end{align*}
\]

\[
\begin{align*}
\beta_2(g_i) &= -\frac{1}{2}\epsilon g_2 - \left[6g_1^3 - 3g_1^2 g_2 - 13g_2^3\right]\frac{1}{12} + \left[549g_1^5 - 537g_1^4 g_2 + 747g_1^3 g_2^2 - 420g_1^2 g_2^3 - 2951g_2^5\right]\frac{1}{648} \\
&\quad - \left[-629856\zeta_3 g_1^7 + 1501722g_1^7 + 1648512\zeta_3 g_1^6 g_2 - 469209g_1^6 g_2 + 2822688\zeta_3 g_1^5 g_2^2 - 1281780g_1^5 g_2^2 \\
&\quad + 1006992\zeta_3 g_1^4 g_2^3 + 1066437g_1^4 g_2^3 - 637632\zeta_3 g_1^3 g_2^4 + 1008810g_1^3 g_2^4 - 563883g_1^2 g_2^5 + 2549232\zeta_3 g_2^7 \\
&\quad - 1449209g_2^7\right]\frac{1}{139968}
\end{align*}
\]

leads to critical exponents, but also to, for example:

\[
N_c(\epsilon) = 2.65 - 4\epsilon^{1/2} + 0.75\epsilon + O(\epsilon^{3/2})
\]
Conformal windows:

Annihilation is with Banks-Zaks-like fixed points near two right edges.

Anything left in $d=5$?
Conclusion:

1) Two examples of fixed-point collisions:
   
a) interacting Luttinger fermions in 3D semiconductors,

b) QED at low N; many others (scalar QED, Potts, QCD (?).....)

2) Characteristic hierarchy of scales due to “walking”; gaps could appear “unnaturally” small

3) Tensor representation of the O(N) models: new IR-stable O(N) fixed points close to d=6.

   Possible non-triviality in d= 4, 5?
Di Pietro et al PRL 2016: neglect of $e^4$ terms gives

1) Fixed points near $d=4$ are at the line $g_+ = 0$

2) Gaussian FP is pinned at $g_- = 0$

3) Critical point goes through it and destabilizes it at

$$1 - 3e_c^2 = 0$$

4) From the leading order beta function for the charge then

$$N_c = (9/4)(4 - d)$$
Yukawa-like field theory for the nematic (IR) critical point:
(Janssen & IH, PRB 2015)

\[ L = L_\psi + L_{\psi\phi} + L_\phi \]

\[ L_\psi = \psi^\dagger (\partial_\tau + \gamma_a d_a (-i \nabla)) \psi, \]

\[ L_{\psi\phi} = g \phi_a \psi^\dagger \gamma_a \psi, \]

\[ L_\phi = \frac{1}{4} T_{ij} \left( -c \partial_\tau^2 - \nabla^2 + r \right) T_{ji} + \lambda T_{ij} T_{jk} T_{ki} + \mathcal{O}(T^4). \]

where the nematic tensorial order parameter is

\[ T_{ij} = \phi_a \Lambda_{a,i,j} \quad \langle \phi_a \rangle = \frac{-g}{r} \langle \psi^\dagger \gamma_a \psi \rangle \]

And \( \Lambda_a \) are the five three dimensional Gell-Mann matrices.
RG flow, close to four (spatial) dimensions:

“B”: “classical” nematic critical point (Priest and Lubensky, 1976)

“F”: new fermionic fixed point