Geometry and Physics of Quantum Hall states

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\( \frac{G}{h} = \frac{1}{R_{xy}} = \begin{cases} \frac{n}{p} \in \mathbb{Q} & \text{fractional QHE} \\ n \in \mathbb{Z}^+ & \text{integer QHE} \end{cases} \)

von Klitzing 1981

Tsui, Stormer, Gossard 1982

Precision measurement of fine structure constant \( \alpha = \frac{e^2}{\hbar c} = \frac{1}{\sqrt{137.035999173(35)}} \)
Integer QHE plateaux are explained by one-particle wave functions (non-interacting)

\[ \psi(z_1, \ldots, z_N) = \det \frac{\Psi_n(z_m)}{n,m=1} \]

\[ \Psi_n = z^n e^{-\frac{B}{4} \|z\|^2} \]

lowest Landau level (LLL) ~ degenerate ground states in strong magnetic field.
Fractional QHE

Strongly interacting (via Coulomb forces) system.

Laughlin 1983: Assign a trial wave function ("state") to each plateau.

\[ \Psi (z_1, ..., z_N) \quad \forall z_n \in \mathbb{C}^N \]

* holomorphic
* vanishing conditions

\[ \Psi = 0 \]
Laughlin state

\[ \Psi(z_1, \ldots, z_N) = c \cdot \prod_{n < m}^{N} (z_n - z_m) \cdot e^{-\frac{B}{4} \sum_{n=1}^{N} |z_n|^2} \]

\[ \beta \in \mathbb{Z}_+ \]

Hall conductance \[ g_h = \frac{1}{\beta} \]

\[ H = \sum_n D_n \overline{D}_n + \sum_{n \neq m} V(z_n, z_m) \]

\[ \overline{D}_n = \frac{\partial}{\partial z_n} + \frac{B}{4} z_n \]
Pfaffian state

Moore-Read 1991:

\[ \psi = \text{Pf} \left( \frac{1}{z_n - z_m} \right) \cdot \prod_{n \neq m} (z_n - z_m)^q \cdot e^{-\frac{B}{4} \sum_n 12n^2} \]

Pfaffian of anti-sym. matrix \( q \in \mathbb{Z}_+ \)

\[ 6_H = \frac{5}{2} \]

\( (q=2) \)

\[ H = \sum_n \mathcal{D}_n \bar{\mathcal{D}}_n + \sum_{n \neq m \neq k} V(z_n, z_m, z_k) \]

\[ \psi = 0 \quad (q=1) \]
How Riemann surfaces show up

* Laughlin's gauge argument (1981)
  
  Change of flux $\Phi$ by $2\pi$
  
  $= \text{transfer of 1 electron}$. 

* TKNN (1982)
  
  Electrons on the lattice $\mathbb{C}/\mathbb{Z}^2$
  
  Bloch bundle over Brillouin zone
  
  Chern number $c_1 = \int d^2k \, d < \mathbf{k} \cdot \mathbf{d}\mathbf{k} >$
Hall conductance from transport on Jacobian

(Aharanov-Bohm, or "solenoid", phases $\Phi_i$)

\[
I_k = i \sum_{i=1}^{2g} \omega_{k,j} \Phi_i, \quad \Phi_j \in [0,2\pi)^{2g} = \text{Jac}(\Sigma)
\]

\[
\omega = \sum_{j=1}^{2g} d\Phi_j \wedge d\Phi_{j+g}
\]

conductance matrix
Adiabatic transport on moduli spaces

Avron, Seiler, Zograf 1995

\[ \sigma_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma} \eta \delta \]
strain-rate

\[ \eta = \frac{1}{4} \mathcal{N}_\phi - \frac{C_H}{24} \gamma(\Sigma) \]
Hall viscosity

deformations of complex structure

\[ C_1 = \eta_H \frac{id\tau \wedge d\bar{\tau}}{(Im \tau)^2} \]

Tokatly, Vignale; N. Read 2008

SK-Wiegmann 2015
Bradlyn-Read 2015
Normalization and large \( N \)

Quantum wave functions shall be normalized

\[
Z = \int_{\mathbb{C}^N} |\Psi_L(z_1, \ldots, z_N)|^2 \prod_{n=1}^N d^2 z_n
\]

\[
= \frac{1}{N!} \int_{\mathbb{C}^N} \exp \left[ -\frac{B}{2} \sum_n z_n^2 + \beta \sum_{n \neq m} \log |z_n - z_m| \right] \prod_{n=1}^N d^2 z_n
\]

2D Coulomb gas partition function.
More generally,

\[ Z = \int_{\mathbb{C}^n} \exp \left\{ -N \sum_{n=1}^{N} V(z_n, \bar{z}_n) + \beta \sum_{n \neq m} \log |z_n - z_m| \right\} \prod_{n=1}^{N} d^2 z_n \]

where

\[ V = \psi(z, \bar{z}) - \frac{1-s}{N} \log \frac{1}{8} g(z, \bar{z}) \]

is the "magnetic potential" and

\[ B = \Delta \psi \]

is the geometric spin (s=1 in pure Coulomb gas).

\[ \text{volume form } \frac{1}{8} g \, d^2 z \text{ on } \mathbb{C} \]

Coulomb gas
Random normal/complex matrices
Beta ensembles
Math result

Thm  Leblé-Serfaty 2017

\[
\log Z = -\beta N^2 I_v(\mu_v) + \frac{\beta}{2} N \log N - NC(\beta) - \\
- N \left(1 - \frac{\beta}{2}\right) \int \mu_v \log \mu_v + \Theta(1)
\]

where  \( I_v = -\iint \log |z-w| \, d\mu(z) \, d\mu(w) + \int V \, d\mu \)

and \( \mu_v \) its unique minimizer ("equilibrium measure")
Physics result

\[ \log Z = -\beta N^2 \mathcal{I}_V(\mu) - N (s - \frac{6}{5}) \int \frac{d \lambda \log \lambda}{\mathcal{O}} \]

\[ \text{Liouville action} \]

\[ \chi_H = 1 - 3 \left( \frac{\phi}{\phi_0} - \frac{2s}{\phi_0} \right)^2 \]

SK 2013
Abanov-Gromov 2014
Can-Laskin-Wiegmann 2014
Ferrari-SK 2014
Bradlyn-Read 2015
SK-Wiegmann 2015

local integrals of $R, B, DR, DB, ...$
Free field representation

CFT w/ background charge & magnetic field

\[ S(\psi) = \frac{i}{2\pi} \int_{\Sigma} \left( \psi \bar{\partial} \psi + \frac{i}{4} \left( \frac{2\pi}{\nu} \right) \psi R \bar{\psi} + \frac{i}{\nu} \psi B \bar{\psi} \right) d^2 z \]

\[ |\Psi_L(z_1, \ldots, z_n)|^2 \approx \langle e^{i \frac{2\pi}{\nu} \psi(z_1)} \ldots e^{i \frac{2\pi}{\nu} \psi(z_n)} \rangle \]

Remainder term:

\[ R_N^{1/N} = \left( S(\psi) + N \log \int_{\Sigma} e^{i \frac{2\pi}{\nu} \psi} d^2 z \right) \]

\[ \approx O \left( \frac{1}{N} \right) \]
Pfaffian state

\[ S = S_b(\psi) + S_+(\psi, \bar{\psi}) \]

\[ S_t = \int t \bar{\psi}\psi + \bar{\psi} \partial \bar{\psi} \]

\[ |\psi_{p,\ell}(z_1, ..., z_n)|^2 \approx \langle \psi \bar{\psi} e^{i\frac{\tau}{2n}} |_{z_1} ... \psi \bar{\psi} e^{i\frac{\tau}{2n}} |_{z_n} \rangle \]

Moore-Read 1991
Magnetoroton excitation mode

Girvin-Macdonald-Platzman 1986

Density wave-type excitations $\rho(k)$, or quasi-hole-quasiparticle pair production with momentum $k$.

$$\varepsilon(k) = \frac{\langle k|H|k \rangle}{\langle k|k \rangle},$$

Static structure factor:

$$S(k) = \langle k|k \rangle = \int \langle \rho(z)\rho(0) \rangle e^{ikz} d^2z$$

$$\rho(z) = \sum_{n=1}^{N} \delta(z-n)$$

We can use large-N expansion for log $Z$ on $S^2$ to compute $S(k)$ on the plane.
\[ < p(z) > = -\frac{1}{N} \frac{\delta}{\delta v(z)} \log Z[\nu] \]

\[ < p(z) p(\omega) > = -\frac{1}{N} \frac{\delta}{\delta v(\omega)} < p(z) > \]

Laughlin \( (\nu = \frac{1}{p}) \)

\[ S(k) = \frac{k^2}{2} + \frac{1-2\nu}{8\nu} k^4 + \frac{(1-3\nu)(3-4\nu)}{96\nu^2} k^6 \]

Fattiani \( (\nu = \frac{1}{q}) \)

\[ S(k) = \frac{k^2}{2} + \frac{1-q}{8q} k^4 + \frac{(1-2q)(2-q)}{64q^2} k^6 \]

leading corr. vanishes for bosonic states, subleading - for fermionic
Laughlin states for genus $g > 0$

Haldane-Rezayi 1985

$\beta$-degeneracy of Laughlin states on torus

Breaking of translation symmetry?

\[ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{torus.png}
\end{array} \]

Wen-Niu 1990

Topological degeneracy on genus $g$

$\beta^g$ FQHE states for filling fraction $\frac{\nu}{g}$ (conjecture).

\[ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{torus2.png}
\end{array} \]

"Topological phases of matter"
Definition of Laughlin states on $\Sigma_g$

$$\Psi(z_1, \ldots, z_N) = \prod_{n<m} (z_n - z_m)^\beta \quad e^{-\frac{B}{4\pi} \sum_n 1/4n^2}$$

$$\bar{D} \Psi = 0 \quad \bar{D}_z = \frac{\partial}{\partial z} + A_z$$

"Shift" or Riemann-Roch then ($\beta = 1$)

$$N = \frac{1}{\beta} N_F + 1 - g$$

* near diagonal $z_n \sim z_m$, $\Psi \propto (z_n - z_m)^\beta$

* $\Psi$ - completely (anti-) symmetric for $\beta \in \text{odd/even}$
Solutions of \( \bar{D} \Psi = 0 \) have \( N_\Phi \) zeroes,

\[
N_\Phi = \beta \left( N-1 + \delta \right)
\]

\[
D = \sum_{\alpha=1}^{N_\Phi} q_\alpha \quad \text{“divisor of zeroes”}
\]

\[
\Psi_L = f(z_1, \ldots, z_N) \prod_{n<\mu} (z_n - z_\mu)^\beta
\]

\[\beta \text{g missing zeroes} \quad \beta \left( N-1 \right) \text{ zeroes}\]
**Theorem**

For $N > g$ the dimension of the vector space of Laughlin states is $\rho^g$.

(Wen-Niu 1990 conjecture)

$$\Psi_r = \Theta \left[ \frac{r^g}{\rho} \right] \left( \rho \sum_{n=1}^{N} z_n - \rho \Delta - \rho D, \beta \tau \right)$$

$$\prod_{n<m} E(z_n, z_m)^{\rho} \cdot \prod_{n=1}^{N} G(z_n)^{\frac{1}{2} N \rho - \rho}$$

* construction of $\rho^g$ functions

* completeness of the basis


w/ D. Zvonkine, to appear
\[ \Psi_r = \Theta \left[ \frac{z}{\rho} \right] \left( \rho \sum_{n=0}^{N} z_n - \beta D, \beta e \right) \cdot \prod_{h<m} E(z_h, z_m)^{\beta} \cdot \prod_{h=m} G(z_h)^{\frac{1}{2} N_\gamma - \beta} \]

Prime form \[ E(z, \bar{y}) \approx \frac{z - \bar{y}}{i dz \wedge d\bar{y}} \]

\( \sigma(z) \sim \text{holomorphic} \quad \frac{dz}{z} \quad \text{form} \)

Abel map: \[ z \mapsto \int \omega_j \in C^3/\Lambda \]

\[ \Sigma \to C/\Lambda \]

\( \omega_j \) - basis of holomorphic 1-forms

\( j = 1, \ldots, g \)

\[ \int \omega_j = \delta j \]

\[ \int \omega_j = \tau j \]

\( \Lambda = \{ m \tau_n \zeta, \quad m, n \in Z^g \} \)

\( N \geq g \)

\( \rho^g \) theta functions
Laughlin states from CFT

$$S(\psi) = \frac{1}{2\pi} \int_{\Sigma} \left( \partial \bar{\psi} \bar{\psi} + \frac{i}{4} \left( \frac{\partial^2}{\partial z^2} \frac{1}{R^2} \right) \psi \bar{\psi} + \frac{i}{2 \pi} \psi \bar{\psi} \right) d^2 z$$

$$\langle e^{i S(\psi_{1})} \cdots e^{i S(\psi_{N})} \rangle \sim \sum_{\epsilon} \sum_{\Psi} (-1)^{\epsilon} \left| \Psi_{\epsilon} (z_{1}, \ldots, z_{N}) \right|^{2}$$

\(\epsilon\) spin structures on \(\Sigma\)

At \(\beta = 1\) this reduces to higher-genus bosonisation \(-1\)

$$\langle e^{i \psi(z_{1})} \cdots e^{i \psi(z_{N})} \rangle = \frac{\det \left( \frac{\partial}{\partial z_{i}} \frac{\partial}{\partial z_{j}} \right) \Psi_{\eta}}{\det <\psi_{\eta}, \psi_{\eta}>^{1/2}} \left| \det \psi_{\eta}(z_{N}) \right|^{2}$$

Verlinde, Verlinde’87, Alvarez-Gaume, Bost, Moore, Nelson, Vafa’97
QHE wave functions are typically degenerate (β^g Laughlin states on genus-g surface) and depend on parameter spaces $\mathcal{M}$ (e.g. moduli space $\mathcal{M}_{g,n}$).

Thus we have a Hilbert bundle $V_{\text{QH}} \to \mathcal{M}$

adiabatic transport:

$$\Psi_r \rightarrow e^{i\phi(\gamma)} U_{\gamma r^{\prime}}(\gamma) \Psi_{r^{\prime}}$$

Berry phase holonomy matrix
Chern-Simons theory

Large-distance "effective" theory of QHE

Witten et al, Wen-Zee, Froehlich, ...

Also: phase $\varphi(\gamma)$

$\varphi(\gamma) = \int \gamma H A dA + \gamma H w dA + \gamma H w dw$

$A$ - magnetic field gauge connection

$w$ - spin connection

$\gamma H, \gamma H, \gamma H$ - Hall conductance, viscosity, central charge
**Definition:** The following are equivalent

* Geometric adiabatic transport is **topological** when holonomy depends only on the topology of the path \( f \) in \( M \), as \( N \to \infty \).
* Berry curvature \( R = c \cdot I + O(1/N) \)
* \( V_{2n} \) is asymptotically projectively flat

**Conjecture** N.Read 2008 (for \( g > 0 \))

CFT-based AH states (e.g. Laughlin and Pfaffian) are asymptotically projectively flat.
Projective flatness of $V_{\text{QH}}$

* Quasi-hole (anyonic) states: $\Psi_{\text{QH}}(z_1, z_2, \ldots, z_n) = \prod (z_i - z_j) \Psi_{\text{ground}}$

For Pfaffian states $V_{\text{QH}}$ has rank 2 on $\mathcal{M}_{0,4}$.

Asymptotic projective flatness:

Bonderson, Gurarie, Nayak 2012
* Laughlin and Pfaffian states are projectively flat on \( \mathcal{M}_{1,1} \)

\[
\mathcal{E}_e \langle \Theta_e(z_1) \ldots \Theta_e(z_n) \rangle = \sum_{\pi} \langle T(z_{\pi}) \Theta_e(z_1) \ldots \Theta_e(z_n) \rangle
\]

\[
\mathcal{E}_e \langle \psi, \psi \rangle_{L^2} = \langle \psi, \nabla^H \psi \rangle_{L^2} \quad T(z) = (\varepsilon \phi)^2 + i \sqrt{\varepsilon} \varepsilon^2 \phi
\]

\[
\nabla^H \psi_z = 0, \quad \nabla^H = 4\pi i N_e \frac{\partial}{\partial z} - \sum_{n \geq 1} \frac{\partial^2}{\partial z_n^2} + 2 \beta (\beta - 1) \sum_{n < m} \mathcal{P} (z_n - z_m)
\]

* Lowest Landau level wave functions on \( \Sigma_{g \geq 1} \)

also form rank-\((N_e + 1 - g)\) vector bundle over \( \mathcal{M}_g \).

It is not proj. flat, unless \( s = \frac{1}{2} \).
The End