Differential transcendence of solutions to elliptic hypergeometric equations through Galois theory\textsuperscript{1}

Carlos E. Arreche\textsuperscript{2}
(joint with Thomas Dreyfus and Julien Roques)

The University of Texas at Dallas

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Introduction

The *elliptic hypergeometric functions* introduced by Spiridonov in the early 2000s are analogues/generalizations of the classical Euler-Gauss hypergeometric functions.

Related to these special functions is an *elliptic hypergeometric equation* that depends on parameters $\varepsilon_1, \ldots, \varepsilon_8$ that satisfy a certain *balancing condition*.

We have shown that if the parameters are “as independent as possible” then no solution of the elliptic hypergeometric equation satisfies an algebraic differential equation with elliptic function coefficients.

The main theoretical tool is *differential Galois theory of difference equations over elliptic curves*. 
Let \( p \in \mathbb{C}^* \) such that \(|p| < 1\), and denote \((z; p)_\infty = \prod_{j \geq 0} (1 - zp^j)\).

We define the \emph{theta function}

\[
\theta(z; p) = (z; p)_\infty (pz^{-1}; p)_\infty \in \text{Mer}(\mathbb{C}^*).
\]

Note that

\[
\theta(z_0; p) = 0 \quad \text{if and only if} \quad z_0 \in p^\mathbb{Z} = \{ p^n \mid n \in \mathbb{Z} \},
\]

and we have the functional equation

\[
\theta(pz; p) = \theta(z^{-1}; p) = -z^{-1} \theta(z; p).
\]
We say that \( f(z) \in \text{Mer}(\mathbb{C}^*) \) is \( p \)-periodic if \( f(pz) = f(z) \).

The field of \( p \)-periodic functions is identified with the field \( \text{Mer}(E) \) of meromorphic functions on the elliptic curve \( E = \mathbb{C}^*/p\mathbb{Z} \).

If \( a_1, \ldots, a_m, b_1, \ldots, b_m \in \mathbb{C}^* \) satisfy the balancing condition

\[
\prod_{j=1}^{m} a_j = \prod_{j=1}^{m} b_j,
\]

the function

\[
c \frac{\prod_{j=1}^{m} \theta(a_j z; p)}{\prod_{j=1}^{m} \theta(b_j z; p)}
\]

\( (c \in \mathbb{C}) \)

is \( p \)-periodic. Any \( p \)-periodic function can be written in this form.
Elliptic gamma functions

Now letting $q \in \mathbb{C}^*$ such that $|q| < 1$ and $p^\mathbb{Z} \cap q^\mathbb{Z} = \{1\}$, we denote $(z; p, q)_\infty = \prod_{j, k \geq 0}(1 - zp^j q^k)$.

The elliptic Gamma function is defined by

$$\Gamma(z; p, q) = \frac{(pq/z; p, q)_\infty}{(z; p, q)_\infty}.$$

Note that

$$\Gamma(pz; p, q) = \theta(z; q)\Gamma(z; p, q)$$

and

$$\Gamma(qz; p, q) = \theta(z; p)\Gamma(z; p, q).$$

These are elliptic analogues of the classical Euler Gamma function $\Gamma(z)$ with $\Gamma(z + 1) = z\Gamma(z)$.

Classical Gauss hypergeometric functions can be defined in terms of the Euler Gamma function (Barnes integral formula).
Elliptic hypergeometric functions

For $t = (t_1, \ldots, t_8) \in (\mathbb{C}^*)^8$ and $p, q \in \mathbb{C}^*$ such that: $|p| < 1, |q| < 1, |t_j| < 1$ for $j = 1, \ldots, 8$, and satisfying the balancing condition

$$\prod_{j=1}^8 t_j = p^2 q^2,$$  \hspace{1cm} (1)

the elliptic hypergeometric function is defined by

$$V(t ; p, q) = \frac{(p; p)_\infty (q; q)_\infty}{4\pi i} \int_{\mathbb{T}} \frac{\prod_{j=1}^8 \Gamma(t_jz^\pm 1; p, q)}{\Gamma(z^\pm 2; p, q)} \frac{dz}{z},$$

where $\mathbb{T}$ denotes the positively-oriented unit circle.

(This is an elliptic version/generalization of the Barnes integral formula in the classical setting).
Reparametrization

Spiridonov introduces the new variables \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_8) \) defined by:

\[
t_6 = c z, \quad t_7 = c / z, \quad \varepsilon_k = \frac{q}{c t_k} \quad \text{for } k = 1, \ldots, 5; \quad \varepsilon_8 = \frac{c}{t_8}; \quad \varepsilon_7 = \frac{\varepsilon_8}{q}; \quad c = \frac{\sqrt{\varepsilon_6 \varepsilon_8}}{p^2}.
\]

One can show the balancing condition \( \prod_{j=1}^{8} \varepsilon_j = p^2 q^2 \) still holds.

**Theorem (Spiridonov)**

The function

\[
f_\varepsilon(z) = \frac{V(q/c \varepsilon_1, \ldots, q/c \varepsilon_5, c z, c / z, c / \varepsilon_8; p, q)}{\Gamma\left(c^2 z^{\pm 1}/\varepsilon_8; p, q\right) \Gamma\left(z^{\pm 1}\varepsilon_8; p, q\right)}
\]

satisfies a second-order linear difference equation over \( \text{Mer}(E) \), where \( E = \mathbb{C}^*/p^\mathbb{Z} \).

This motivates the study of \( f_\varepsilon(z) \) through (differential) Galois theory of difference equations over elliptic curves.
Galois theory (philosophy)

A *Galois theory* associates to an equation (polynomial, differential, difference..., over a “well-understood” base field $K$) a *Galois group* that encodes properties of the solutions (over $K$).

**Group Theory**

⇓

Galois Theory

**Relations among Solutions**

Computing Galois groups leads directly to computation of relations (over $K$) among the solutions to the corresponding equations.

“Large” Galois group $\iff$ “few” relations among solutions.
The base $\sigma\delta$-field of elliptic functions

As before, we let $p, q \in \mathbb{C}^*$ such that:

$$|p| < 1, \quad |q| < 1, \quad \text{and} \quad p^\mathbb{Z} \cap q^\mathbb{Z} = \{1\}.$$

The last condition means that $q \pmod{p^\mathbb{Z}}$ is of infinite order in the abelian group $E = \mathbb{C}^*/p^\mathbb{Z}$.

**Base field:** $K = \mathcal{M}er(E)$, the field of meromorphic functions on $E$.

**Difference operator:** The automorphism $\sigma : f(z) \mapsto f(qz)$.

**Differential operator:** The invariant derivation $\delta$ on $E$ is $\delta = z \frac{d}{dz}$.

With this, $K$ is a $\sigma\delta$-field: $\sigma \circ \delta = \delta \circ \sigma$. 
Differential Galois theory of difference equations

Let $K$ be a $\sigma\delta$-field such that $K^\sigma = \{c \in K \mid \sigma(c) = c\} = C$ is $\delta$-algebraically-closed, and consider a linear difference equation

$$a_n\sigma^n(y) + a_{n-1}\sigma^{n-1}(y) + \cdots + a_1\sigma(y) + a_0y = 0, \quad (2)$$

where $a_n, \ldots, a_0 \in K$ and $a_na_0 \neq 0$.

To (2) is associated a $\sigma\delta$-$K$-algebra $R$, generated by a $C$-basis of solutions $y_1, \ldots, y_n \in R$ and their iterates under $\sigma$ and $\delta$.

The $\sigma\delta$-Galois group

$$\text{Gal}_{\sigma\delta}(R/K) := \{\gamma \in \text{Aut}_{K\text{-alg}}(R) \mid \gamma \circ \sigma = \sigma \circ \gamma, \gamma \circ \delta = \delta \circ \gamma\}$$

gets identified with a linear differential algebraic group in $\text{GL}_n(C)$.

\[3\text{And also det}(\sigma^{i-1}(y_j))^{-1}, \text{ where } 1 \leq i, j \leq n.\]
Linear differential algebraic groups

Definition
If $C$ is a $\delta$-field, we write $C^\delta := \{ c \in C \mid \delta(c) = 0 \}$. A linear $\delta$-algebraic group is a subgroup of $\text{GL}_n(C)$ defined by algebraic differential equations in the matrix entries.

Examples:
- algebraic groups over $C$;
- algebraic groups over $C^\delta$;

Let $\mathcal{L} = \sum_{i=0}^{n} c_i \delta^i$ with $c_n, \ldots, c_0 \in C$.

- $\{ \beta \in \mathbb{G}_a(C) \mid \mathcal{L}(\beta) = 0 \}$;
- $\{ \alpha \in \mathbb{G}_m(C) \mid \mathcal{L}\left( \frac{\delta(\alpha)}{\alpha} \right) = 0 \}$.

Theorem (Cassidy)
Every $\delta$-algebraic subgroup of $\mathbb{G}_a(C)$ or $\mathbb{G}_m(C)$ is as above.
Main Result: sufficient Galois-theoretic criteria for differential transcendence

[Under mild conditions on the otherwise arbitrary $\sigma\delta$-field $K$.]

Theorem (A.-Dreyfus-Roques)
Let $f \neq 0$ be a solution of

$$\sigma^2(f) + a\sigma(f) + bf = 0,$$

where $a, b \in K$ and $b \neq 0$. Assume that:

- There is no $u \in K$ such that $\sigma(u)u + au + b = 0$.
- There are no $c_0, \ldots, c_n \in \mathbb{C}$ with $c_n \neq 0$ and $h \in K$, such that

$$c_n\delta^n\left(\frac{\delta b}{b}\right) + \cdots + c_0\frac{\delta b}{b} = \sigma(h) - h.$$

Then $f$ is differentially transcendental over $K$. 
The elliptic hypergeometric equation

Theorem (Spiridonov)

The function $f_\varepsilon(z)$ satisfies

$$A(z)(\sigma(y) - y) + A(z^{-1})(\sigma^{-1}(y) - y) + \nu y = 0,$$

where

$$A(z) = \frac{1}{\theta(z^2; p)\theta(qz^2; p)}\prod_{j=1}^{8} \theta(\varepsilon_jz; p), \quad \nu = \prod_{j=1}^{6} \theta(\varepsilon_j\varepsilon_8/q; p).$$

- It follows from the balancing condition $\prod_{j=1}^{8} \varepsilon_j = p^2 q^2$ that the coefficients $A(z)$, $A(z^{-1}) \in \text{Mer}(E) = K$.

- Hence, (3) is equivalent to a second-order linear difference equation over the base $\sigma\delta$-field $K$ (after applying $\sigma$).

- Because the coefficients $A(z)$ and $A(z^{-1})$ are given in terms of theta functions, we have complete knowledge of their divisors.
Differential transcendence result

Recall that $\varepsilon_1, \ldots, \varepsilon_8, p, q$ satisfy the balancing condition

$$\prod_{j=1}^{8} \varepsilon_j = p^2 q^2. \quad (4)$$

We say that $\varepsilon_1, \ldots, \varepsilon_8, p, q$ are “as independent as possible” if every multiplicative relation among $\varepsilon_1, \ldots, \varepsilon_8, p, q$ is induced from (4). That is, whenever

$$\varepsilon_1^{n_1} \cdots \varepsilon_8^{n_8} p^{m_1} q^{m_2} = 1$$

with $n_1, \ldots, n_8, m_1, m_2 \in \mathbb{Z}$, we have that $n_1 = \cdots = n_8 = \alpha \in \mathbb{Z}$ and $m_1 = m_2 = -2\alpha$.

**Theorem (A.-Dreyfus-Roques)**

*If $\varepsilon_1, \ldots, \varepsilon_8, p, q$ are as independent as possible then every solution to the corresponding elliptic hypergeometric equation is differentially transcendental over $\operatorname{Mer}(E)$.***
Proving differential transcendence

To prove differential transcendence, we verified the conditions of our Main Result assuming that \( \varepsilon_1, \ldots, \varepsilon_8, p, q \) are “as independent as possible”.

- Earlier work of Dreyfus-Roques provides criteria to decide (non-)existence of solutions \( u \in \text{Mer}(E) \) to Riccati equation

\[
\sigma(u)u + au + b = 0,
\]

depending on the divisors of \( a, b \in \text{Mer}(E) \).

- The non-existence of a telescoper \( 0 \neq \mathcal{L} \in C[\delta] \) and certificate \( h \in \text{Mer}(E) \) such that

\[
\mathcal{L} \left( \frac{\delta(b)}{b} \right) = \sigma(h) - h
\]

is also proved by analyzing the divisor of \( b \in \text{Mer}(E) \).
Sketch of proof: Main Result (1/2)

We know a priori that one of the following three cases occurs for the $\sigma\delta$-Galois group $G$.

1. $G$ is conjugate to a group of upper-triangular matrices. This happens if and only if there exists a solution $u \in K$ to the Riccati equation $\sigma(u)u + au + b = 0$.

2. $G$ is conjugate to a subgroup of

$$\begin{cases}
\begin{pmatrix}
\alpha & 0 \\
0 & \beta
\end{pmatrix} & \alpha, \beta \in \mathbb{C}^\times
\end{cases} \cup \begin{cases}
\begin{pmatrix}
0 & \gamma \\
\mu & 0
\end{pmatrix} & \gamma, \mu \in \mathbb{C}^\times
\end{cases}.$$ 

3. $G$ contains $\text{SL}_2(\mathbb{C}^\delta)$.

No solutions to Riccati equation implies that $G$ is irreducible (i.e., we are not in case 1).
No telescoper/certificate for \( \mathcal{L}(\frac{\delta(b)}{b}) = \sigma(h) - h \) implies that \( \det(G) = \mathbb{G}_m(C) \), which in turn implies that \( G \) is one of the following groups

\[
\begin{align*}
\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \mid \alpha, \beta \in \mathbb{C}^\times \} \cup \{ \begin{pmatrix} 0 & \gamma \\ \mu & 0 \end{pmatrix} \mid \gamma, \mu \in \mathbb{C}^\times \};
\end{align*}
\]

- \( \mathbb{C}^\times \cdot \text{SL}_2(\mathbb{C}^\delta) \);
- \( \text{GL}_2(\mathbb{C}) \).

In any of these cases, \( G \) is sufficiently large to guarantee that any one \( f \neq 0 \) to the difference equation must be differentially transcendental over \( K \).
Summary

- We applied differential Galois theory of difference equations over elliptic curves to show that any solution of the elliptic hypergeometric equation must be differentially transcendental, provided that $\varepsilon_1, \ldots, \varepsilon_8, p, q$ are as independent as possible.

- This hypothesis rules out the functions $f_\varepsilon(z)$ introduced by Spiridonov as motivation to study the elliptic hypergeometric equation in the first place.

- Hence, our Main Result remains silent about the elliptic hypergeometric functions $V(t;p,q)$, as well as the $f_\varepsilon(z)$.

- In future work we hope to prove differential transcendence of $f_\varepsilon(z)$ under the assumption that $\varepsilon_1, \ldots, \varepsilon_7, p, q$ are as independent as possible subject to the balancing condition

$$\varepsilon_1 \cdots \varepsilon_6 \varepsilon_7^2 = p^2 q.$$