Source identities for relativistic models of CMS type

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Elliptic integrable systems, special functions and quantum field theory
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Non-relativistic models
Trigonometric A-type: Sen’s model
A generalization of the Sutherland model:

$$\mathcal{H}_N^A(X; m) = -\sum_{J=1}^{N} \frac{1}{m_J} \frac{\partial^2}{\partial X_J^2} + \sum_{1 \leq J < K \leq N} \gamma_{J,K} \cdot \sin(X_J - X_K)^{-2}$$

where

$$\gamma_{J,K} = \lambda(m_J + m_K)\left(\lambda m_J m_K - 1\right)$$

(D. Sen 1996)

Pertinent eigenfunction:

$$\Phi_0^A(X, m) = \prod_{1 \leq J < K \leq N} \sin(X_J - X_K)^{\lambda m_J m_K}$$

satisfying the eigenvalue equation

$$(\mathcal{H}_N^A(X; m) - \mathcal{E}_0) \Phi_0^A(X; m) = 0, \quad \mathcal{E}_0 = \mathcal{E}_0(N, m, \lambda).$$

"=Source identity for non-relativistic trigonometric A-type"
Trigonometric $A$-type: Sen’s model

A generalization of the Sutherland model:

$$H^A_N(X; m) = - \sum_{J=1}^{N} \frac{1}{m_J} \frac{\partial^2}{\partial X_J^2} + \sum_{1 \leq J < K \leq N} \gamma_{J,K} \cdot \sin(X_J - X_K)^{-2}$$

where

$$\gamma_{J,K} = \lambda (m_J + m_K) (\lambda m_J m_K - 1)$$

(D. Sen 1996)

- Specialization 1: Let $N = N \in \mathbb{Z}_{>0}$ and $m_J = 1$ for all $J$, $H^A_N \rightarrow H^A_N(x; \lambda)$ and $\Phi^A_0$ reduces to the groundstate eigenfunction for the Sutherland Schrödinger operator $\psi_0(x; \lambda)$.

- Specialization 2: Let $N = N + M$ and $m_J = 1$ for $J = 1, \ldots, N$ and $m_J = -1$ for $J = N + 1, \ldots, M$, $H^A_N \rightarrow H^A_N(x; \lambda) - H^A_M(y; \lambda)$ and $\Phi^A_0 \rightarrow$ "kernel function"

- Specialization 3: Let $N = N + \tilde{M}$ and $m_J = 1$ for $J = 1, \ldots, N$ and $m_J = +1/\lambda$ for $J = N + 1, \ldots, \tilde{M}$, $H^A_N \rightarrow H_N(x; \lambda) + \lambda H^A_{\tilde{M}}(\tilde{y}; +1/\lambda)$ and $\Phi^A_0 \rightarrow$ "dual-kernel function".
Trigonometric $A$-type: Sen’s model

A generalization of the Sutherland model:

$$
\mathcal{H}_N^A(X; m) = -\sum_{J=1}^{N} \frac{1}{m_J} \frac{\partial^2}{\partial X_J^2} + \sum_{1 \leq J < K \leq N} \gamma_{J,K} \cdot \sin(X_J - X_K)^{-2}
$$

where

$$
\gamma_{J,K} = \lambda (m_J + m_K)(\lambda m_J m_K - 1)
$$

(D. Sen 1996)

Specialization 4: Let $N = N + \tilde{N}$ and $m_J = 1$ for $J = 1, \ldots, N$ and $m_J = -1/\lambda$ for $J - N = 1, \ldots, \tilde{N}$,

$$
\mathcal{H}_N^A \to H_{N,\tilde{N}}(x; \tilde{x}; \lambda) = H_N(x; \lambda) - \lambda H_{\tilde{N}}(\tilde{x}; 1/\lambda)
$$

$$
+ 2(1 - \lambda) \sum_{j=1}^{N} \sum_{k=1}^{\tilde{N}} \sin(x_j - \tilde{x}_k)^{-2}
$$

becomes the differential operator defining the deformed CMS model of type $A$ (Chalykh, Feigin, and Veselov 1998, Sergeev 2001 & 2002)
Trigonometric $A$-type: Sen’s model
A generalization of the Sutherland model:

$$
\mathcal{H}^A_N(\mathbf{X} \,; \, \boldsymbol{m}) = - \sum_{J=1}^{N} \frac{1}{m_J} \frac{\partial^2}{\partial X_J^2} + \sum_{1 \leq J < K \leq N} \gamma_{J,K} \cdot \sin(\mathbf{X}_J - \mathbf{X}_K)^{-2}
$$

where

$$
\gamma_{J,K} = \lambda (m_J + m_K) (\lambda m_J m_K - 1)
$$

(D. Sen 1996)

Specialization 5: Let $\mathcal{N} = N + \tilde{N} + M + \tilde{M}$ and

$$
m_J = \begin{cases} 
1, & J = 1, \ldots, N \\
-1/\lambda, & J - N = 1, \ldots, \tilde{N} \\
-1, & J - N - \tilde{N} = 1, \ldots, M \\
+1/\lambda, & J - N - \tilde{N} - M = 1, \ldots, \tilde{M}
\end{cases}
$$

$\Rightarrow$ Kernel function identity for a pair of deformed CMS operators
Trigonometric \textit{BC}-type:

Generalization of the \textit{BC}-type CMS model

\[
\mathcal{H}_{N}^{BC}(X; m) = \sum_{J=1}^{N} \frac{1}{m_J} \left(-\frac{\partial^2}{\partial X_j^2} + \sum_{\nu=0}^{1} d_{\nu,J}(d_{\nu,J} - 1) \sin(X_j + \frac{1}{2}\omega_{\nu})^{-2}\right)
+ \sum_{1\leq J<K\leq N} \gamma_{J,K} \left(\sin(X_J + X_K)^{-2} + \sin(X_J - X_K)^{-2}\right)
\]

where \(\omega_0 = 0\), \(\omega_1 = \pi\), and \(d_{\nu,J} = m_J d_{\nu} + \frac{1}{2} \lambda m_J (m_J - 1)\) (Hallnäss and Langmann 2010)

Pertinent eigenfunction:

\[
\Phi_{0}^{BC} = \prod_{J=1}^{N} \prod_{\nu=0}^{1} \sin(X_J + \frac{1}{2}\omega_{\nu})^{d_{\nu,J}} \prod_{1\leq J<K\leq N} \left[\sin(X_J - X_K) \sin(X_J + X_K)\right]^{\lambda m_J m_K}
\]

satisfying the eigenvalue equation \((\mathcal{H}_{N}^{BC}(X; m) - \mathcal{E}_{0}^{BC})\Phi_{0}^{BC}(X; m) = 0\)
Trigonometric $BC$-type:

Generalization of the $BC$-type CMS model

$$
\mathcal{H}^{BC}_{N}(X; m) = \sum_{J=1}^{N} \frac{1}{m_J} \left( -\frac{\partial^2}{\partial X_j^2} + \sum_{\nu=0}^{1} d_{\nu,J} (d_{\nu,J} - 1) \sin(X_j + \frac{1}{2}\omega_{\nu})^{-2} \right)
+ \sum_{1\leq J<K\leq N} \gamma_{J,K} \left( \sin(X_J + X_K)^{-2} + \sin(X_J - X_K)^{-2} \right)
$$

where $\omega_0 = 0$, $\omega_1 = \pi$, and $d_{\nu,J} = m_J d_{\nu} + \frac{1}{2}\lambda m_J (m_J - 1)$

(Hallnäs and Langmann 2010)

Specializations $\Rightarrow$ groundstate eigenfunction and eigenvalue identity, deformed $BC$-type differential operator, i.e.

$$
H^\ast_{N,\tilde{N}} = H_N(x; \{d_{\nu}\}, \lambda) - \lambda H_{\tilde{N}}(\tilde{x}; \{(1 + \lambda - 2d_{\nu})/2\lambda\}, 1/\lambda)
+ 2(1 - \lambda) \sum_{j=1}^{N} \sum_{k=1}^{\tilde{N}} \sin(x_j - \tilde{x}_k)^{-2} + \sin(x_j + \tilde{x}_k)^{-2},
$$

and ‘groundstate’ eigenvalue identity, and various kernel function identities

(Hallnäs and Langmann, Sergeev and Veselov)
Elliptic cases

Elliptic generalizations? Yes!
Replace potential with \( \varphi(x), \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \),

Elliptic \( A \)-type:

- Source identity under balancing condition \( \lambda \sum J m_J = 0 \)
  (Langmann 2010)

- Specializations \( \Rightarrow \) deformed elliptic \( A \)-type differential operator and
  ‘groundstate’ eigenvalue identity \( (\lambda N - \tilde{N} = 0) \), and various kernel function
  identities
  (Chalykh, Feigin, and Veselov, Khodarinova, Langmann)

Elliptic \( BC \)-type:

- Source identity under balancing condition \( 2\lambda \sum J m_J + \sum \nu d_\nu = 0 \)
  (Langmann and Takemura 2012)

- Specializations \( \Rightarrow \) groundstate eigenvalue identity, deformed elliptic CMS
  operator and ‘groundstate’ eigenvalue identity \( (2\lambda N - \tilde{N} + \sum \nu d_\nu = 0) \), and
  various kernel function identities
  (Langmann and Takemura 2012)
Relativistic models
Elliptic $A$-type

Generalization of the elliptic Ruijsenaars model

$$
S^A_N(X; m) = \sum_{J=1}^{N} \vartheta_1(i\lambda m J \beta) \left( \prod_{K \neq J} f_+(X_J - X_K; m_J, m_K)^{1/2} \right) \\
\times \exp (+i \frac{\beta}{m_J} \frac{\partial}{\partial X_J}) \left( \prod_{K \neq J} f_-(X_J - X_K; m_J, m_K)^{1/2} \right)
$$

(FA, M. Hallnäs, and E. Langmann 2014)

where $\beta > 0$ is the “relativistic deformation” parameter and

$$
f_{\pm}(x; m, m') = \frac{\vartheta_1(x \mp i\xi(m, m') \beta \mp i\lambda m' \beta)}{\vartheta_1(x \mp i\xi(m, m') \beta)}
$$

with $\xi(m, m') = (m - m')(\lambda mm' - 1)/4mm'$
Elliptic $A$-type

Generalization of the elliptic Ruijsenaars model

$$S_N^A(X; m) = \sum_{J=1}^{N} \vartheta_1(i \lambda m_J \beta) \left( \prod_{K \neq J} f_+(X_J - X_K; m_J, m_K)^{\frac{1}{2}} \right)$$

$$\times \exp(+i \frac{\beta}{m_J} \frac{\partial}{\partial X_J}) \left( \prod_{K \neq J} f_-(X_J - X_K; m_J, m_K)^{\frac{1}{2}} \right)$$

(FA, M. Hallnäs, and E. Langmann 2014)

Pertinent eigenfunction of the form

$$\Phi_0^A(X; m) = \prod_{1 \leq J < K \leq N} \phi(X_J - X_K; m_J, m_K)$$

for $m \in \{1, -1, +1/\lambda, -1/\lambda\}$ satisfying $S_N^A(X; m) \Phi_0^A(X; m) = 0$ under the balancing condition $\lambda \sum_J m_J = 0$. 
Elliptic $A$-type

Generalization of the elliptic Ruijsenaars model

$$S^A_N(X; m) = \sum_{J=1}^{N} \vartheta_1(i\lambda m_J \beta) \left( \prod_{K \neq J} f_+(X_J - X_K; m_J, m_K)^{\frac{1}{2}} \right) \times \exp(+i \frac{\beta}{m_J} \frac{\partial}{\partial X_J}) \left( \prod_{K \neq J} f_-(X_J - X_K; m_J, m_K)^{\frac{1}{2}} \right)$$

(FA, M. Hallnäs, and E. Langmann 2014)

Specializations $\Rightarrow$ kernel function identities for pairs of Ruijsenaars operators (Ruijsenaars 2006 and Komori, Noumi, and Shiraishi 2009)

Also, Chalykh-Feigin-Sergeev-Veselov type generalization of Ruijsenaars model
Elliptic deformed $A$-type

Generalization of the elliptic Ruijsenaars model

$$S_{N,\tilde{N}}^A(x, \tilde{x}; \lambda, \beta) = \sum_{j=1}^{N} \left( A_j^+ \right)^{1/2} e^{i\beta \frac{\partial}{\partial x_j}} \left( A_j^- \right)^{1/2} - \frac{\vartheta_1(i\beta)}{\vartheta_1(i\lambda\beta)} \sum_{k=1}^{\tilde{N}} \left( B_k^+ \right)^{1/2} e^{-i\lambda\beta \frac{\partial}{\partial \tilde{x}_k}} \left( B_k^- \right)^{1/2}$$

with ‘groundstate’ eigenfunction

$$\Psi_{N,\tilde{N}}^A(x, \tilde{x}; \lambda, \beta) = \frac{\Psi_N^A(x; \lambda, \beta) \Psi_{\tilde{N}}^A(\tilde{x}; 1/\lambda, \lambda\beta)}{\prod_{j}^{N} \prod_{k}^{\tilde{N}} \left[ \vartheta_1(x_j - \tilde{x}_k + i\frac{1}{2}(\lambda - 1)\beta) \vartheta_1(x_j - \tilde{x}_k - i\frac{1}{2}(\lambda - 1)\beta) \right]^{1/2}}$$

with $\Psi_N^A$ the groundstate for the Ruijsenaars model (balancing condition $\lambda N - \tilde{N} = 0$).

Gauge transform w.r.t. $\Psi_{N,\tilde{N}}^A$, i.e. $A_{N,\tilde{N}}^A = (\Psi_{N,\tilde{N}}^A)^{-1} \circ S_{N,\tilde{N}}^A \circ \Psi_{N,\tilde{N}}^A$ yields a more familiar form:
Elliptic deformed $A$-type

Generalization of the elliptic Ruijsenaars model

$$\mathcal{A}_{N,\tilde{N}}^A(x,\tilde{x};\lambda,\beta) = \sum_{j=1}^{N} A_j^+ \exp(i\beta \frac{\partial}{\partial x_j}) - \frac{\vartheta_1(i\beta)}{\vartheta_1(i\lambda\beta)} \sum_{k=1}^{\tilde{N}} B_k^+ \exp(-i\lambda\beta \frac{\partial}{\partial \tilde{x}_k})$$

with coefficients

$$A_j^+ = \prod_{j' \neq j}^{N} \left( \frac{\vartheta_1(x_j - x_{j'}) + i\lambda\beta}{\vartheta_1(x_j - x_{j'})} \right) \prod_{k=1}^{\tilde{N}} \left( \frac{\vartheta_1(x_j - \tilde{x}_k + \frac{1}{2}i(\lambda - 1)\beta)}{\vartheta_1(x_j - \tilde{x}_k + \frac{1}{2}i(\lambda + 1)\beta)} \right)$$

$$B_k^+ = \prod_{k' \neq k}^{\tilde{N}} \left( \frac{\vartheta_1(\tilde{x}_k - \tilde{x}_{k'}) - i\beta}{\vartheta_1(\tilde{x}_k - \tilde{x}_{k'})} \right) \prod_{j=1}^{N} \left( \frac{\vartheta_1(\tilde{x}_k - x_j + \frac{1}{2}i(\lambda - 1)\beta)}{\vartheta_1(\tilde{x}_k - x_j - \frac{1}{2}i(\lambda + 1)\beta)} \right)$$
Elliptic deformed A-type

Generalization of the elliptic Ruijsenaars model

\[ A_{N,\tilde{N}}^A (x, \tilde{x}; \lambda, \beta) = \sum_{j=1}^{N} A^+_j \exp(i\beta \frac{\partial}{\partial x_j}) - \frac{\vartheta_1(i\beta)}{\vartheta_1(i\lambda \beta)} \sum_{k=1}^{\tilde{N}} B^+_k \exp(-i\lambda \beta \frac{\partial}{\partial \tilde{x}_k}) \]

and obtain \( (A_{N,\tilde{N}}^A (x, \tilde{x}; \lambda, \beta) - A_{M,\tilde{M}}^A (-y, -\tilde{y}; \lambda, \beta)) K_{N,\tilde{N},M,\tilde{M}} (x, \tilde{x}, y, \tilde{y}) = 0 \), for (balancing condition \( \lambda(N - M) - (\tilde{N} - \tilde{M}) = 0 \))

\[ K_{N,\tilde{N},M,\tilde{M}} (x, \tilde{x}, y, \tilde{y}) = \left( \prod_{j=1}^{N} \prod_{K=1}^{M} \frac{G(x_j - y_k - i\frac{1}{2} \lambda \beta; \beta)}{G(x_j - y_k + i\frac{1}{2} \lambda \beta; \beta)} \right) \left( \prod_{j=1}^{\tilde{N}} \prod_{k'=1}^{\tilde{M}} \vartheta_1(x_j - \tilde{y}_{k'}) \right) \]

\[ \times \left( \prod_{j'=1}^{\tilde{N}} \prod_{k=1}^{M} \vartheta_1(\tilde{x}_{j'} - y_k) \right) \left( \prod_{j'=1}^{\tilde{N}} \prod_{k'=1}^{\tilde{M}} \frac{G(\tilde{x}_{j'} - \tilde{y}_{k'} - i\frac{1}{2} \beta; \lambda \beta)}{G(\tilde{x}_{j'} - \tilde{y}_{k'} + i\frac{1}{2} \beta; \lambda \beta)} \right) \]

where \( G(x; \alpha) \) is the elliptic Gamma function: \( G(x + i\frac{1}{2} \alpha; \alpha)/G(x - i\frac{1}{2} \alpha; \alpha) = \text{const.} \cdot \vartheta_1(x) \).
Elliptic $BC$-type

Generalization of the van Diejen model

$$S_{N}^{BC}(X; m) = \sum_{\varepsilon=\pm} \sum_{J=1}^{N} \vartheta_{1}(i\lambda m_{J}\beta)(V_{J}^{\varepsilon})^{\frac{1}{2}} \exp(-\varepsilon i \frac{\beta}{m_{J}} \frac{\partial}{\partial X_{J}})(V_{J}^{-\varepsilon})^{\frac{1}{2}} + V^{0}(X; m)$$

for $m \in \{1, -1/\lambda, -1, +1/\lambda\}^{N}$,

where

$$V^{\pm} = \frac{\prod_{\nu=0}^{7} \vartheta_{1}(\pm X_{J} - ig_{\nu,J}\beta)}{\vartheta_{1}(\pm 2X_{J})\vartheta_{1}(\pm 2X_{J} - i\beta/m_{J})} \prod_{\delta=\pm} \prod_{K \neq J} f_{\pm}(X_{J} + \delta X_{K}; m_{J}, m_{K})$$

for $g_{\nu,J} = g_{\nu} - \lambda(m_{J} - 1)/4 + (1/m_{J} - 1)/4$ and $V^{0}$ an elliptic function.
Elliptic $BC$-type

Generalization of the van Diejen model

$$S_{N}^{BC}(X; m) = \sum_{\varepsilon=\pm} \sum_{J=1}^{N} \vartheta_{1}(i\lambda m_{J} \beta)(\mathcal{V}_{J}^{\varepsilon})^{1/2} \exp(-\varepsilon i \beta \frac{\partial}{m_{J} \partial X_{J}})(\mathcal{V}_{J}^{\varepsilon})^{1/2} + \mathcal{V}^{0}(X; m)$$

for $m \in \{1, -1/\lambda, -1, +1/\lambda\}^{N}$,

Pertinent eigenfunction of the form

$$\Phi_{0}^{BC}(X; m) = \left(\prod_{J=1}^{N} \psi(X_{J}; m_{J})\right) \left(\prod_{\varepsilon, \delta=\pm} \prod_{1 \leq J < K \leq N} \phi(\varepsilon X_{J} + \delta X_{K}; m_{J}, m_{K})\right)$$

for $m \in \{1, -1, +1/\lambda, -1/\lambda\}^{N}$ satisfying $S_{N}^{BC}(X; m)\Phi_{0}^{BC}(X; m) = 0$ under the balancing condition $2\lambda \sum_{J} m_{J} + \sum_{\nu=0}^{7} g_{\nu} - 2(\lambda + 1) = 0$. 
Elliptic $BC$-type

Generalization of the van Diejen model

$$S_{N}^{BC}(X; m) = \sum_{\varepsilon=\pm}^{N} \vartheta_1(i\lambda m J \beta)(\mathcal{V}_J^\varepsilon)^{\frac{1}{2}} \exp(-\varepsilon i \frac{\beta}{m J} \frac{\partial}{\partial X_J})(\mathcal{V}_J^{-\varepsilon})^{\frac{1}{2}} + \mathcal{V}^0(X; m)$$

for $m \in \{1, -1/\lambda, -1, +1/\lambda\}^N$,

Specializations $\Rightarrow$ groundstate eigenvalue identity and kernel function identities for pairs of van Diejen operators (different couplings)

(Ruijsenaars 2009 and Komori, Noumi, and Shiraishi 2009)

Also, Chalykh-Feigin-Sergeev-Veselov type generalization of the van Diejen model
Elliptic deformed $BC$-type

Generalization of the van Diejen model: Conjugate with ‘groundstate’ function

\[\Psi_{N,\tilde{N}}^{BC} = \frac{\Psi_N^{BC}(x; \{g_\nu\}, \lambda, \beta) \Psi_{\tilde{N}}^{BC}(\tilde{x}; \{ (\lambda + 1 - 2g_\nu) / 2\lambda \}, 1/\lambda, \lambda \beta)}{\prod_{j=1}^N \prod_{k=1}^{\tilde{N}} \prod_{\varepsilon, \delta = \pm} \vartheta_1(\varepsilon x_j + \delta \tilde{x}_k + i \frac{1}{2} (\lambda - 1) \beta^{1/2})}\]

i.e. \[A_{N,\tilde{N}}^{BC} = (\Psi_{N,\tilde{N}}^{BC})^{-1} \circ S_{N,\tilde{N}}^{BC} \circ \Psi_{N,\tilde{N}}^{BC}\]

then

\[A_{N,\tilde{N}}^{BC} = \sum_{\varepsilon = \pm} \sum_{j=1}^N V_j^\varepsilon e^{-i\varepsilon \beta \frac{\partial}{\partial x_j}} - \frac{\vartheta_1(i\beta)}{\vartheta_1(i\lambda \beta)} \sum_{k=1}^{\tilde{N}} V_k^\varepsilon e^{+i\varepsilon \lambda \beta \frac{\partial}{\partial \tilde{x}_k}} + V^0\]
Elliptic deformed $BC$-type

Generalization of the van Diejen model:

$$A_{N,\tilde{N}}^{BC} = \sum_{\epsilon=\pm} \sum_{j=1}^{N} V^{\epsilon} e^{-i\epsilon\beta \frac{\partial}{\partial x_j}} - \frac{\vartheta_1(i\beta)}{\vartheta_1(i\lambda\beta)} \sum_{k=1}^{\tilde{N}} \tilde{V}^{\epsilon} e^{+\epsilon\lambda\beta \frac{\partial}{\partial \tilde{x}_k}} + V^0$$

with coefficients ($\tilde{g}_\nu = (\lambda + 1 - 2g_\nu)/2\lambda$)

$$V^{\pm}_j = U(\pm x_j; \{g_\nu\}, \beta) \prod_{\delta=\pm} \prod_{j' \neq j} f^{\pm}(x_j + \delta x_{j'}; 1, 1) \prod_{k=1}^{\tilde{N}} f^{\pm}(x_j + \delta \tilde{x}_k; 1, -1/\lambda)$$

$$\tilde{V}^{\pm}_k = U(\mp \tilde{x}_k; \{\tilde{g}_\nu\}, \lambda\beta) \prod_{\delta=\pm \pm} \prod_{k' \neq k} f^{\pm}(\tilde{x}_k + \delta \tilde{x}_{k'}; -1/\lambda, -1/\lambda) \prod_{j=1}^{N} f^{\pm}(\tilde{x}_k + \delta x_j; -1/\lambda, 1)$$

$$U(x; \{g_\nu\}, \beta) = \prod_{\nu=0}^{7} \vartheta_1(x - ig_\nu \beta)\vartheta_1(2x)\vartheta_1(2x - i\beta), \quad f^{\pm}(x; m, m') = \frac{\vartheta_1(x \mp i\frac{1}{2}\lambda(m + m')\beta)}{\vartheta_1(x \mp i\frac{1}{2}\lambda(m - m')\beta)}$$
Elliptic deformed $BC$-type

Generalization of the van Diejen model:
Let $2\lambda(N - 1) - 2(\tilde{N} + 1) + \sum_{\nu} g_{\nu} = 0$ for simplicity, then

$$\mathcal{A}_{N,\tilde{N}}^{BC} = \sum_{\varepsilon = \pm} \sum_{j=1}^{N} V_{j}^{\varepsilon} \left( e^{-i\varepsilon \beta \frac{\partial}{\partial x_j}} - 1 \right) - \frac{\vartheta_1(i\beta)}{\vartheta_1(i\lambda\beta)} \sum_{k=1}^{\tilde{N}} \tilde{V}_{k}^{\varepsilon} \left( e^{+i\varepsilon \lambda \beta \frac{\partial}{\partial \tilde{x}_k}} - 1 \right)$$

with coefficients $(\tilde{g}_{\nu} = (\lambda + 1 - 2g_{\nu})/2\lambda)$

$$V_{j}^{\pm} = U(\pm x_j; \{g_{\nu}\}, \beta) \prod_{\delta = \pm} \prod_{j' \neq j} f_{\pm}(x_j + \delta x_{j'}; 1, 1) \prod_{k=1}^{\tilde{N}} f_{\pm}(x_j + \delta \tilde{x}_k; 1, -\frac{1}{\lambda})$$

$$\tilde{V}_{k}^{\pm} = U(\mp \tilde{x}_k; \{\tilde{g}_{\nu}\}, \lambda\beta) \prod_{\delta = \pm} \prod_{k' \neq k} f_{\pm}(\tilde{x}_k + \delta \tilde{x}_{k'}; -\frac{1}{\lambda}, -\frac{1}{\lambda}) \prod_{j=1}^{N} f_{\pm}(\tilde{x}_k + \delta x_{j}; -\frac{1}{\lambda}, 1)$$

$$U(x; \{g_{\nu}\}, \beta) = \prod_{\nu=0}^{7} \frac{\vartheta_1(x - ig_{\nu}\beta)}{\vartheta_1(2x)\vartheta_1(2x - i\beta)}$$

$$f_{\pm}(x; m, m') = \frac{\vartheta_1(x \mp i\frac{1}{2} \lambda(m + m')\beta)}{\vartheta_1(x \mp i\frac{1}{2} \lambda(m - m')\beta)}$$
Elliptic deformed $BC$-type

Generalization of the van Diejen model:

$$
\mathcal{A}_{N,\tilde{N}}^{BC} = \sum_{\varepsilon=\pm} \sum_{j=1}^{N} V_j^\varepsilon e^{-i\varepsilon\beta \frac{\partial}{\partial x_j}} - \frac{\vartheta_1(i\beta)}{\vartheta_1(i\lambda\beta)} \sum_{k=1}^{\tilde{N}} \tilde{V}_k^\varepsilon e^{+i\lambda\beta \frac{\partial}{\partial \tilde{x}_k}} + V^0
$$

and obtain

$$(\mathcal{A}_{N,\tilde{N}}^{BC}(\mathbf{x}, \tilde{\mathbf{x}}, \{g_\nu\}, \lambda, \beta) - \mathcal{A}_{M,\tilde{M}}^{BC}(\mathbf{y}, \tilde{\mathbf{y}}, \{\frac{1}{2}(\lambda + 1) - g_\nu\}, \lambda, \beta)) K_{N,\tilde{N},M,\tilde{M}}^{BC} = 0$$

for (balancing condition $2\lambda(N - M - 1) - 2(\tilde{N} - \tilde{M} + 1) + \sum_\nu g_\nu = 0$)

$$
K_{N,\tilde{N},M,\tilde{M}}^{BC} = \left( \prod_{j=1}^{N} \prod_{k=1}^{M} \prod_{\varepsilon, \delta = \pm} G(\varepsilon x_j + \delta y_k - i\frac{1}{2} \lambda \beta; \beta) \right) \left( \prod_{j=1}^{N} \prod_{k'=1}^{\tilde{M}} \prod_{\delta = \pm} \vartheta_1(x_j + \delta \tilde{y}_{k'}) \right)
$$

$$\times \left( \prod_{j'=1}^{\tilde{N}} \prod_{k=1}^{M} \prod_{\varepsilon, \delta = \pm} \vartheta_1(\tilde{x}_{j'} + \delta y_k) \right) \left( \prod_{j'=1}^{\tilde{N}} \prod_{k'=1}^{\tilde{M}} \prod_{\varepsilon, \delta = \pm} G(\varepsilon \tilde{x}_{j'} + \delta \tilde{y}_{k'} - i\frac{1}{2} \beta; \lambda\beta) \right)$$
Summary:

- Constructed generalizations of the first difference operator of Ruijsenaars and van Diejen models and found an exact eigenfunction of these operators.
- Specializing the 'mass' parameters yields deformed generalizations of Ruijsenaars and van Diejen models and their kernel function identities.

Outlook:

- Higher order difference operators of deformed elliptic models.
- Special functions related to the deformed elliptic models.
- Application of relativistic deformed models to other areas of mathematics and physics.
- As always: A way of removing the balancing condition.
Thank you!