Elliptic Stirling and Lah numbers

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based on joint work with

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Outline

1. Classical Stirling and Lah numbers
2. Carlitz’ $q$-Stirling numbers
3. Garsia and Remmel’s $q$-Lah numbers
4. Elliptic Stirling numbers
5. Elliptic Lah numbers
Classical Stirling and Lah numbers

We recall some facts about the Stirling numbers of the second and the first kind.

We denote the falling factorials by

\[ x \overset{n}{\ldots}_{1} := \begin{cases} x(x-1)\ldots(x-n+1) & \text{if } n = 1, 2, \ldots, \text{ or } \text{if } n = 0, \\ 1 & \text{if } n = 0. \end{cases} \]

Similarly, we denote the raising factorials by

\[ x \overset{n}{\ldots}^1 := \begin{cases} x(x+1)\ldots(x+n-1) & \text{if } n = 1, 2, \ldots, \text{ or } \text{if } n = 0, \\ 1 & \text{if } n = 0. \end{cases} \]
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\bar{x}^n := \begin{cases} 
  x(x + 1) \ldots (x + n - 1) & \text{if } n = 1, 2, \ldots, \\
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\end{cases}
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The **Stirling numbers of the second kind**, \( S(n, k) \), are defined as the following connection coefficients:

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As is easy to see (use $x = (x - k) + k$), the $S(n, k)$ satisfy the following recursion:

$$S(n, 0) = \delta_{n,0},$$
$$S(n, k) = 0 \quad \text{for } k > n,$$
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By replacing $x$ by $-x$ in the defining relation (and multiplying both sides by $(-1)^n$), we immediately obtain

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$$s(n, 0) = \delta_{n,0},$$
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We also have

$$\bar{x}^n = \sum_{k=0}^{n} (-1)^{n-k} s(n, k) x^k.$$
Clearly,

\[ \sum_{k=l}^{n} S(n, k) s(k, l) = \delta_{n,l}, \]

or

\[ (S(n, k))^{-1}_{n,k \in \mathbb{N}_0} = (s(k, l))_{k,l \in \mathbb{N}_0}. \]
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\[ (S(n, k))_{n,k \in \mathbb{N}_0}^{-1} = (s(k, l))_{k,l \in \mathbb{N}_0}. \]

Explicit formula:

\[ S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n \]

\[ = \frac{k^n}{k!} {}_nF_{n-1} \left( \begin{array}{c} 1-k, 1-k, \ldots, 1-k \end{array} ; 1; -k, \ldots, -k \right). \]
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Thus, the Lah numbers are the result of **convolution** of the Stirling numbers of the two different kinds:

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Explicit formula:

$$L(n, k) = \binom{n}{k} \frac{(n-1)!}{(k-1)!}.$$ 

This follows easily from the $y = n - 1$ case of the Chu-Vandermonde summation formula

$$\binom{x + y}{n} = \sum_{k=0}^{n} \binom{x}{k} \binom{y}{n-k}.$$
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The Lah numbers further satisfy the recursion formula

\[
L(n + 1, k) = L(n, k - 1) + (n + k)L(n, k).
\]
Combinatorial interpretations:

- The Stirling numbers of the second kind, $S(n, k)$, count the number of possibilities to partition a set of $n$ elements into $k$ blocks.
- The (unsigned) Stirling numbers of the first kind, $c(n, k) = (-1)^{n-k} s(n, k)$, count the number of permutations of an $n$ element set with exactly $k$ cycles.
- The Lah numbers, $L(n, k)$, count the number of possibilities to arrange a set of $n$ elements into $k$ nonempty linearly ordered subsets.

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In order to define Carlitz’ $q$-Stirling numbers of the second and the first kind, we need some $q$-notation. For (complex) $q \neq 1$, we define the $q$-number of $x$ by

$$[x]_q := 1 - q^x.$$ (Clearly, $\lim_{q \to 1}[x]_q = x$, by de L'Hôpital's rule.)

We denote the $q$-falling factorials by

$$[x]_q^n := \begin{cases} [x]_q [x-1]_q \ldots [x-n+1]_q & \text{if } n = 1, 2, \ldots, \ 1 & \text{if } n = 0. \end{cases}$$

Similarly, we denote the $q$-raising factorials by

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As is easy to see (use \([x]_q = q^k[x - k]_q + [k]_q\)), they satisfy the following recursion:

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S_q(n, 0) = \delta_{n,0},
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S_q(n, k) = 0 \quad \text{for } k > n,
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By replacing \(x\) by \(-x\), and \(q\) by \(1/q\), in the above defining relation (and multiplying both sides by \((-q)^{-n}\)), we immediately obtain

\[
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\[ \sum_{k=l}^n S_q(n, k) s_q(k, l) = \delta_{n,l}, \]

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**Explicit formula** by Carlitz:
\[
S_q(n, k) = \frac{1}{[k]_q!} \sum_{j=0}^{k} (-1)^j q^{(j)} \binom{k}{j} \frac{[k - j]_q^n}{[k]_q}
\]
\[
= \frac{[k]_q^n}{[k]_q!} n^{\phi}_{n-1} \left[ q^{1-k}, q^{1-k}, \ldots, q^{1-k}, q^{-k}, \ldots, q^{-k} ; q, q^{k-n} \right].
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\[ = \frac{[k]^n_q}{[k] q!} n^\phi n^{-1} \left[ q^{1-k}, q^{1-k}, \ldots, q^{1-k} \right]^{q^{-k}}_{q^{-k}, \ldots, q^{-k}; q, q^{k-n}}. \]

**Combinatorial interpretations** of Carlitz’ \( q \)-Stirling numbers of the second kind were given by

- **S. Milne** – restricted growth functions,
- **Garsia & Remmel** – rook configurations,

and others.
The \( q \)-Lah numbers \( L_q(n, k) \), can be defined as the following connection coefficients:

\[
[x^n_q] = \sum_{k=0}^{n} L_q(n, k) [x^k_q].
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Explicit formula:  

$$L_q(n, k) = q^{k(k-1)} \begin{bmatrix} n \end{bmatrix}_q \frac{[n-1]_q!}{[k-1]_q!},$$

where $[0]_q! = 1$ and $[m]_q! = [m]_q[m - 1]_q!$, and $\begin{bmatrix} n \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$.
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This can be shown to follow from the recursion formula

$$L_q(n+1, k) = q^{n+k-1}L_q(n, k-1) + [n+k]_qL_q(n, k).$$
Elliptic Stirling numbers

In order to define our new elliptic Stirling numbers of the second and the first kind, we need to introduce some notation.

Let $|p| < 1$.

(Modified Jacobi) theta functions:

$$\theta(z; p) := \left( z, \frac{p}{z}; p \right) \prod_{j=0}^{\infty} \left( 1 - p^j z \right) \left( 1 - \frac{p^j}{z} \right).$$

There holds

$$\theta(z; 0) = (1 - z).$$

Compact notation:

$$\theta(z_1, \ldots, z_m; p) := \theta(z_1; p) \cdots \theta(z_m; p).$$
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**Addition formula:**

\[
\begin{align*}
\theta(zw, z/w, uv, u/v; p) - \theta(zv, z/v, uw, u/w; p) &= \frac{u}{w} \theta(wv, y/w, zu, z/u; p).
\end{align*}
\]
Now, for (complex) $a, b, q, p$, $q \neq 1$, $p < 1$, we define the elliptic number of $x$ (or the $a, b; q, p$-number of $x$) by

$$[x]_{a,b;q,p} := \frac{\theta(q^x, aq^x, bq^2, a/b; p)}{\theta(q, aq, bq^{1+x}, aq^{x-1}/b; p)}.$$
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\]

The elliptic number of \(x\) indeed extends the \(q\)-number of \(x\), which is obtained as the following limit:

\[
\lim_{b \to 0} \left( \lim_{a \to 0} \left( \lim_{p \to 0} [x]_{a,b;q,p} \right) \right) = [x]_q.
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The elliptic numbers satisfy a number of nice identities, for instance

\[
[x + 1]_{a,b;q,p} - [x]_{a,b;q,p} = \frac{\theta(aq^{1+2x}, bq, bq^2, a/b, a/bq; p)}{\theta(aq, bq^{1+x}, bq^{2+x}, aq^x/b, aq^{x-1}/b; p)} q^x
\]

(the expression on the r.h.s. is very-well-poised), among others, by the addition formula for theta functions.
We denote the elliptic falling factorials by 
\[
\left[ x \right]_{a, b; q, p} := \left\{ \begin{array}{ll}
\left[ x \right]_{a, b; q, p} & \\
\left[ x - 1 \right]_{aq^2, bq; q, p} & \\
\left[ x - n + 1 \right]_{aq^{2n-2}, bq^{n-1}; q, p} & \text{if } n = 1, 2, \ldots \\
1 & \text{if } n = 0.
\end{array} \right.
\]

Similarly, we denote the elliptic raising factorials by 
\[
\left[ x \right]_{a, b; q, p} := \left\{ \begin{array}{ll}
\left[ x \right]_{a, b; q, p} & \\
\left[ x + 1 \right]_{aq^{-2}, bq^{-1}; q, p} & \\
\left[ x + n - 1 \right]_{aq^{2n-2}, bq^{n-1}; q, p} & \text{if } n = 1, 2, \ldots \\
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\[
[x]_{a,b; q, p}^n :=
\begin{cases}
[x]_{a,b; q, p}[x - 1]_{aq^2, bq; q, p} \cdots [x - n + 1]_{aq^{2n-2}, bq^{n-1}; q, p} & \text{if } n = 1, 2, \ldots , \\
1 & \text{if } n = 0.
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Similarly, we denote the elliptic raising factorials by

\[
[x]^n_{a,b;q,p} := \begin{cases} 
[x]_{a,b;q,p}[x + 1]_{aq^{-2},bq^{-1};q,p} \ldots [x + n - 1]_{aq^{-2n},bq^{1-n};q,p} & \text{if } n = 1, 2, \ldots, \\
1 & \text{if } n = 0.
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Our elliptic Stirling numbers of the second kind, $S_{a,b;q,p}(n,k)$, are defined as the following connection coefficients:

$$[x]_{a,b;q,p}^n = \sum_{k=0}^{n} S_{a,b;q,p}(n,k) [x]_{a,b;q,p}^k.$$
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As one can verify, the $S_{a,b; q, p}(n, k)$ satisfy the following recursion:

$$S_{a,b; q, p}(n, 0) = \delta_{n,0},$$ $$S_{a,b; q, p}(n, k) = 0 \quad \text{for } k > n,$$ $$S_{a,b; q, p}(n + 1, k) = \frac{\theta(aq^{2k-1}, bq, bq^2, a/b, a/bq; p)}{\theta(aq, bq^k, bq^{1+k}, aq^{k-1}/b, aq^{k-2}/b; p)} q^{k-1}$$ $$\times S_{a,b; q, p}(n, k - 1) + [k]_{a,b; q, p} S_{a,b; q, p}(n, k).$$
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Analogous to the \( q \)-case, one can also deduce a corresponding identity involving the elliptic raising factorials (which we omit here).
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Again, analogous to the $q$-case, one can also deduce a corresponding identity involving the elliptic raising factorials (which we omit here).
Clearly,

\[ \sum_{k=l}^{n} S_{a,b;q,p}(n, k) s_{a,b;q,p}(k, l) = \delta_{n,l}, \]

or

\[ (S_{a,b;q,p}(n, k))_{n,k\in\mathbb{N}_0}^{-1} = (s_{a,b;q,p}(k, l))_{k,l\in\mathbb{N}_0}. \]
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Explicit expressions:
\[
S_{a,b; q,p}(n, 0) = \delta_{n,0},
\]
\[
S_{a,b; q,p}(n, 1) = 1 - \delta_{n,0},
\]
\[
S_{a,b; q,p}(n, 2) = [2]_{a,b; q,p}^{n-1} - 1,
\]
\[
S_{a,b; q,p}(n, 3) = [2]_{aq^2, bq; q,p}^{-1} - 1
\]
\[
\times \left( [3]_{a,b; q,p}^{n-1} - [2]_{aq^2, bq; q,p} aq^5, bq^2, a/b; p) + \theta(aq^3, bq^4, aq^2/b; p) q \right).
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\]
\[
S_{a,b;q,p}(n, 3) = [2]_{aq^2,bq;q,p}^{-1} \left( [3]_{a,b;q,p}^{n-1} - [2]_{aq^2,bq;q,p} [2]_{a,b;q,p}^{n-1} + \frac{\theta(aq^5, bq^2, a/b; p)}{\theta(aq^3, bq^4, aq^2/b; p)} q \right).
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So far, we were not able to find any explicit formula for \( S_{a,b;q,p}(n, k) \) for general \( n \) and \( k \).
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\[
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So far, we were not able to find any explicit formula for \(S_{a,b;q,p}(n, k)\) for general \(n\) and \(k\).

In particular, it appears that \(S_{a,b;q,p}(n, k)\) cannot be written as a multiple of an elliptic hypergeometric series.
Elliptic Lah numbers

The elliptic Lah numbers $L_{a, b}; q, p(n, k)$ can be defined as the following connection coefficients:

$$\left[ x^k \right]^a, b; q, p = \sum_{j=0}^{n} L_{a, b}; q, p(n, j) \left[ x^j \right]^a, b; q, p.$$ 

Thus, the elliptic Lah numbers are the result of convolution of the elliptic Stirling numbers of the two different kinds:

$$L_{a, b}; q, p(n, k) = \sum_{j=0}^{k} S_{a, b}; q, p(n, j) \left[ x^k \right]^a, b; q, p.$$ 

The elliptic Lah numbers satisfy the recursion formula

$$L_{a, b}; q, p(n+1, k) = \theta(aq^{2k-1}, bq, bq^2, aq-1/n-1/b, aq/n/b) \times q^{n+k-1}L_{a, b}; q, p(n, k-1) + \left[ n+k \right]aq^{2n}L_{a, b}; q, p(n, k).$$
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The elliptic Lah numbers, in general, do **not** factorize in closed form (contrary to the classical case) but they do factorize nicely in the $p \to 0$, $b \to 0$ limiting case:
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Let $L_{a; q}(n, k) = \lim_{b \to 0} (\lim_{p \to 0} L_{a, b; q, p}(n, k))$. 

Explicit formula:

$$L_{a; q}(n, k) = q^{\binom{k}{2}} - \binom{n}{2} - n(k - 1) \frac{n!}{(n-k)!} q^n a^k - 1^n q^{n+k} \left( a\frac{n}{2} - n + 1 \right) q^{n+k} a^{k-1}.$$ 

It is easy to see that in the $a \to \infty$ limit we recover the $q$-Lah number $L_{q}(n, k)$.
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**Explicit formula:**

\[
L_{a;q}(n, k) = q^{\binom{k}{2} - \binom{n}{2} - n(k-1)} \left[\begin{array}{c} n \\ k \end{array}\right] \frac{[n-1]_q!}{[k-1]_q!} \frac{(aq^{k-n+1}; q)_{n+k}}{(aq^{3-2n}; q^2)_n(aq^2; q^2)_k},
\]

where $(z; q)_m = (1 - z)(1 - zq) \cdots (1 - zq^{m-1})$. 
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Final remarks:


Combinatorial interpretations of the above elliptic Stirling and Lah numbers have been given in terms of rook theory [M.S. & Meesue Yoo, Electronic J. Combin. 24(1) (2017), #P1.31].

Rather than using elliptic weights, one can also give weight-dependent generalizations of the above elliptic Stirling and Lah numbers [Zs´ofia R. Keresk´ényin´e Balogh & M.S., unpublished].

The weights can be specialized to yield, e.g., symmetric function analogues of the Stirling and Lah numbers.
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