On the Rank of the Reduced Density Operator for the Laughlin State and Symmetric Polynomials

Licentiate Thesis in Theoretical Physics

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Abstract

One effective tool to probe a system revealing topological order is to bipartition the system in some way and look at the properties of the reduced density operator corresponding to one part of the system. In this thesis we focus on a bipartition scheme known as the particle cut in which the particles in the system are divided into two groups and we look at the rank of the reduced density operator. In the context of fractional quantum Hall physics it is conjectured that the rank of the reduced density operator for a model Hamiltonian describing the system is equal to the number of quasi-hole states. Here we consider the Laughlin wave function as the model state for the system and try to put this conjecture on a firmer ground by trying to determine the rank of the reduced density operator and calculating the number of quasi-hole states. This is done by relating this conjecture to the mathematical properties of symmetric polynomials and proving a theorem that enables us to find the lowest total degree of symmetric polynomials that vanish under some specific transformation referred to as clustering transformation.

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Contents

Introduction and Outline

1.1 Introduction

From personal experience, we know that matter is found in three different states or phases, solid, liquid, and gas. For example, a bunch of water molecules can be found in all these three states as ice, liquid water, and steam. Cooling an amount of liquid water down to its freezing temperature transforms it to solid ice. The liquid phase transforms to the solid phase and it is said that a phase transition has occurred. Although a full description of these states also needs quantum physics, traditionally these three states belong to a category called classical states of matter. Another important phase of matter is the ferromagnetic phase, already known from the time of Ancient Greek in the form of permanent magnets. At high temperatures, the magnetic moments of a magnetic material, are disordered. By cooling down the material, these magnetic moments align if the temperature falls below a certain temperature, called the Curie temperature. Other examples of phase of matter are the superfluid phase, the superconducting phase and the liquid crystal phases.

What distinguishes these phases from each other is their internal structure, or in other words, their internal order. Consider a single atomic gas as an example. The interaction between atoms is almost zero and, therefore, each atom is moving unrelated to the motion of the other atoms. Thus, one can say that the gaseous state is a very disordered one and that the gas is symmetric under a translation with respect to any vector of an arbitrary magnitude and direction. At low temperature, the kinetic energy of atoms is much lower, and the interaction of atoms is more important. So the the motion of individual atoms influence each other and a regular pattern known as crystal or lattice is formed. This lattice is symmetric with respect to only those translations whose corresponding vector is an integer multiple of the lattice vector. That is, the continuous translational symmetry is broken to a discrete translational symmetry. In the case of ferromagnetism mentioned above, one notes that for high temperature the spins of the electrons in a piece of material are randomly aligned so that the average magnetic moment is zero. In this case, the system has a continuous rotational symmetry known as $SO(3)$ symmetry. But below

the Curie temperature the magnetic moments of the system align, giving rise to a non-zero magnetic moment and the ferromagnetic state emerges. In this ferromagnetic state, the rotational symmetry is broken.

By considering the relation between the internal order and the symmetries of phases of matter, Russian physicist Lev Landau developed a theory, now known as Landau's theory of phase transitions, to explain all these different phases and the transitions between them. The main idea underlying his theory is the idea of symmetry breaking. Roughly speaking, this idea expresses that in a phase transition from some disordered phase to a more ordered one, some symmetry is lost. In this theory, the notion of the local order parameter plays a crucial role. In the ordered phase, the order parameter takes a finite value, while its value is zero in the disordered phase. In the case of ferromagnetism, the magnetization plays the role of the local order parameter.

Landau's theory is very successful in explaining phases and the transitions between them. However, Landau's theory does not capture all phases of matter. As is explained in Chapter 2 in more detail, the German physicist Klaus von Klitzing found that at low temperatures, and in a strong magnetic field, the Hall resistance of a two-dimensional electron gas, instead of varying smoothly proportional to the strength of the magnetic field as one expects classically, actually changed in steps and showed a pattern of plateaus [vKDP80]. It turned out that the Hall conductance σ_H of these plateaus can, to very high accuracy, be expressed as a product of an integer times e^2/h, the fundamental unit of conductance, where e is the charge of the electron, and h is the Planck constant. This phenomenon is known as the integer quantum Hall effect (IQHE). von Klitzing received the 1985 Nobel Prize in physics for this discovery. Two years later, Horst L. Störmer and Daniel Tsui at Bell labs—by doing the same kind of experiment on a much cleaner sample, and at a temperature of about $1\,\mathrm{K}$, and a magnetic field of about $30\,\mathrm{T}$—discovered a new plateau [TSG82]. But this time, the Hall conductance could be described as a fractional number times e^2/h, namely $\sigma_H = e^2/(3h)$. This phenomenon is known as the fractional quantum Hall effect (FQHE).

As is described in Chapter 2, the IQHE was explained theoretically soon after its discovery by considering the physics of a free electron moving in two dimensions, in the presence of a strong magnetic field. This simplicity stems from the fact that in this case the Coulomb interaction between electrons can be ignored, at least in the first approximation. In contrast, in the case of FQHE, Coulomb interactions are important and the system is a strongly correlated system. Interestingly, the internal order corresponding to a fractional quantum Hall (FQH) system, does not allow for a description in terms of Landau's theory of phase transitions. Instead, it was realized that the FQH system is a completely new state of matter.

In 1983, Robert Laughlin from Stanford University came up with a way to explain the FQHE [Lau83]. His idea was based on introducing an approximate trial wave function that captured the important aspects of the physics of a

system with the fractional Hall conductance $\sigma_H = e^2/(3h)$ observed in Störmer and Tsui's experiment. Laughlin's trial wave function, that explains the $1/m$ fractional quantum Hall effect up to a normalization constant, is

$$\Psi_m(z_1, \ldots, z_N) = \prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)^m \, \exp\Big(-\frac{1}{4l_B^2} \sum_{k=1}^{N} |z_k|^2 \Big), \qquad (1.1)$$

where z_1 till z_N are the electron coordinates in the complex plane and l_B is a constant of length dimension. At this stage it is good to know that the wave function above is an approximate eigenstate of the real Hamiltonian, that is, the Hamiltonian with Coulomb interaction as its interaction term. For small system sizes, namely systems with only a few electrons, numerical calculations confirmed more than 99% overlap between Laughlin's trial wave function (1.1) and the ground state wave function for the real Hamiltonian. Despite of being an approximate eigenstate for the real Hamiltonian, Laughlin wave function is the exact solution of a model Hamiltonian, in the following sense. There is actually a mathematical expression for the interaction term for which the Laughlin state is the exact ground state[§] [Hal83]. The Hamiltonian with this mathematical expression as its interaction term is called the model Hamiltonian.

Some illustrations are in order here. Consider some "total" Hamiltonian H_{total} defined by

$$H_{\text{total}} = \lambda \, H_{\text{model}} + (1 - \lambda) \, H_{\text{real}}, \qquad (1.2)$$

where H_{model} is the model Hamiltonian mentioned above, H_{real} is the real Hamiltonian, and $0 \leqslant \lambda \leqslant 1$ is a real parameter. Numerical investigations confirm that if one continuously varies the parameter λ from zero to one, one does not encounter any phase transition. This justifies why the model Hamiltonian and Laughlin's wave function can be used to study interesting physical properties of FQH systems. The Laughlin wave function is explained in more detail in Chapter 2.

The FQHE cannot be described in terms of Landau's theory of phase transitions. This is because the FQH states do not break any symmetry, and there is no local order parameter. Instead, one says that the FQH states have *topological order* [Wen95]. One manifestation of topological order is that on higher genus surfaces, the phase shows a ground state degeneracy. For instance, on a sphere, the Laughlin state is unique, while on the torus, it has an m fold degeneracy [Wen95]. Thus, a topologically ordered phase is sensitive to the topology of the surface it lives on. Therefore, topologically ordered phases have intricate non-local properties.

As the lines above try to motivate, the physics of a system with topological order, like a FQH system, is very rich and it is important to study the non-local

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[§]This interaction basically enforces that the wave function should vanish at least as an mth power (instead of a first power, which is necessary because of the Pauli principle) when two electrons are at the same location.

nature of these kind of systems. One way to probe systems with topological order, is to partition the system into two subsystems in some way and look at different properties of reduced density operator corresponding to each part of the system. In general, one can consider all the eigenvalues of the reduced density operator but in this thesis we consider only the rank of the reduced density operator.

In the FQH context, different ways of bipartitioning the total Hilbert space $\mathcal{H}$, namely, the *orbital cut*, the *real-space cut*, and the *particle cut* have been proposed [ZHSR07, HZS07, LH08, DRR12, SCR$^+$12, RSS12]. In this thesis we deal with the particle cut scheme in which one attaches numbers to N particles (electrons) in the system and declares the particles numbered 1 till N_A to belong to subsystem A and the remaining particles numbered $N_A + 1$ till N to belong to subsystem B. Numerical investigations provides evidence that the following conjecture holds. The content of this conjecture is explained in more detail in Section 5.3.

Conjecture 1.1 (**Rank Saturation Conjecture**). *The rank of the reduced density operator corresponding to a particle cut of a model state, like a Laughlin or a Moore–Read state [MR91], is equal to the number of quasi-hole states in an appropriate number of flux quanta, that is, the number of ground states of the model Hamiltonian in an appropriate magnetic field.*

The main goal of this thesis is to put this conjecture on a stronger footing by considering a special case of this conjecture. We consider a FQH system in a pure Laughlin state $\Psi_m(z_1, \ldots, z_N)$, as the model state, and try to determine the rank of the reduced density operator associated with a particle cut of the system and compare this number with the number of independent quasi-hole wave functions.

As is shown in Section 5.3, the conjecture above is satisfied for the special case $m = 1$. For m greater than one, we were not able to find a rigorous proof but we made some progress. We realized that Conjecture 1.1 is equivalent to the following mathematically formulated conjecture:

Conjecture 1.2. *There is no non-zero symmetric polynomial in mN variables with degree, in each variable, less than $N + 1$ that vanishes under the transformation that clusters the mN variables in m groups, with N variables in each group, and identifies the variables in each group.*

The transformation mentioned in the conjecture above is referred to as the clustering transformation and it is formally defined in Section 5.1. The content of Conjecture 1.2 becomes clear during subsequent chapters. This observation led us to study the properties of symmetric polynomials, and in particular their properties under clustering transformation. It turned out that proving Conjecture 1.2 is very hard, and we did not succeed completely. However, we were able to prove that there are no non-zero symmetric polynomials in mN

variables with *total degree* less than $N + 1$, that vanish under the clustering transformation. In addition, we found a full characterization of the symmetric polynomials that vanish under the clustering transformation.

1.2 Outline

The thesis is organized as follows. Chapter 2 gives a very short introduction to classical and quantum Hall effects. It also introduces the Laughlin wave function and gives the physical motivation behind this wave function. Chapter 3 is a review of the reduced density operator and the statement of the Schmidt decomposition theorem. Here weak Schmidt decomposition is introduced as well. Chapter 4 is a review of the basics of the theory of symmetric functions. Chapter 5 provides a candidate for a weak Schmidt decomposition of the Laughlin state Ψ_m. It also provides an upper bound for the rank of the reduced density operator for a FQH system modeled by the Laughlin state when the system is subjected to a particle cut. The last section of this chapter is devoted to a description of the content of the Conjecture 1.1. Chapter 6, which is the main contribution of this thesis, introduces a new set of generators for the algebra of symmetric polynomials and probe some of their interesting properties. Chapters 5 and 6 are based on the accompanied paper [GEA15].

Classical and Quantum Hall Effects

This chapter starts with a brief presentation of classical and quantum Hall effects and continues by revisiting the well-known problem of determining the energy levels of an electron in a magnetic field, known as Landau levels. Then it introduces the Laughlin wave function and explains how Laughlin came to this particular form of a wave function to describe the fractional quantum Hall effect. For more detailed calculations the reader can refer to any standard textbook on quantum Hall effect [K.J07, Eza08, Yos02].

2.1 Classical Hall Effect

In 1879 Edwin Hall, an American physics graduate student at Johns Hopkins University, observed that if a thin strip of a conducting material that carries a longitudinal electric current is subjected to a perpendicular uniform magnetic field $\boldsymbol{B}$, a transverse voltage appears. Classically this is easy to explain. Consider a thin strip of a conducting material lying on the $x^1 O x^2$ plane carrying a longitudinal electric current along the positive direction of the $O x^2$ axis, that is, the electrons are moving in the opposite direction. When a uniform magnetic field $\boldsymbol{B}$ in the positive direction of the $O x^3$ axis is turned on, the electrons in the strip are affected by the Lorentz force $\boldsymbol{F} = e \boldsymbol{v} \times \boldsymbol{B}$ that lies on the plane of the strip perpendicular to its length. Here e ($e < 0$) is the electric charge of electron and $\boldsymbol{v}$ is its velocity. Under this force electrons accumulate on one longitudinal edge of the strip, giving rise to a transverse voltage. This continues until the magnetic force on the electrons is balanced by the force exerted on them due to the so-called *Hall electric field* $\boldsymbol{E}_H$ created by the transverse voltage. At this point, electrons flow along the strip without being disturbed by any transverse acceleration. Therefore,

$$e \boldsymbol{v} \times \boldsymbol{B} + e \boldsymbol{E}_H = 0. \tag{2.1}$$

In this context, the transverse resistivity ρ_{12} is known as *Hall resistivity* and it is denoted by ρ_H. To see how classical physics relates the Hall resistivity to the magnitude B of the magnetic field, consider the current density

$$\boldsymbol{j} = n e \boldsymbol{v}, \tag{2.2}$$

where n is the number-density of electrons. Two components $E_{H,1}$ and $E_{H,2}$ of the Hall electric field are related to components of $\boldsymbol{j}$ through the resistivity tensor $\rho = [\rho_{\mu\nu}]_{2\times 2}$ according to

$$E_{H,\mu} = \sum_{\nu=1}^{2} \rho_{\mu\nu} j_{\nu}. \tag{2.3}$$

For this problem it is straight forward to see that

$$\rho = \frac{B}{n|e|} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{2.4}$$

Therefore, classical physics predicts that ρ_{12} is proportional to the magnitude of the magnetic field according to the following equation:

$$\rho_{12} = \frac{B}{n|e|}, \tag{2.5}$$

and the *longitudinal resistivities* ρ_{11} and ρ_{22} vanish. By taking the inverse of the resistivity tensor in Equation (2.4) the conductivity tensor σ is found to be

$$\sigma = \frac{n|e|}{B} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.6}$$

In contrast, by doing measurements on a silicon MOSFET (metal-oxide-semiconductor field effect transistor), von Klitzing found that the Hall resistivity does not follow the classical predictions [vKDP80]. It was revealed that increasing the magnetic field on some intervals does not affect the Hall resistance ρ_H so that on these intervals the Hall resistance remains constant. In other words the graph of ρ_H versus the magnetic field B shows plateaus. But of course, as in the classical case, on these plateaus the longitudinal resistance is zero as is shown in Figure 2.1. It is also measured to a very high accuracy that the Hall resistance ρ_H on each plateau obeys the simple relation

$$\rho_H = \frac{1}{\nu} \frac{h}{e^2}, \tag{2.7}$$

where h is the Planck constant and ν is a rational number and, consequently,

$$\sigma_H = \nu \frac{e^2}{h}. \tag{2.8}$$

Therefore, on the plateaus, Equations (2.4) and (2.6) are corrected for the following ones

$$\rho = \frac{1}{\nu} \frac{h}{e^2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma = \nu \frac{e^2}{h} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.9}$$

The experiments also revealed that the number ν is either an integer or a simple fraction with an odd denominator. As mentioned in Chapter 1, the former case is called the integral quantum Hall effect (IQHE) and the latter case is called the fractional quantum Hall effect (FQHE).
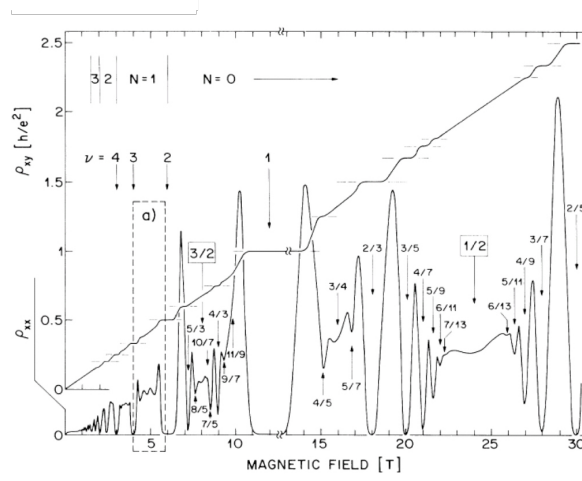
Figure 2.1: FQHE. This picture is taken from [Wil13]

2.2 Landau Levels and Quantum Hall Effects

The corner stone of theoretical understanding of the integral and fractional quantum Hall effects is the quantum treatment of a free electron in a magnetic field. Consider an electron of mass m_e and charge e that is subjected to a uniform strong[§] magnetic field B along the positive Ox^3 direction. Also assume that the electron is somehow confined to move in the x^1Ox^2 plane[¶]. The corresponding Hamiltonian is

$$H = \frac{1}{2m_e}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2,\tag{2.10}$$

where c is the speed of light, $\boldsymbol{p} = -\mathrm{i}\hbar\boldsymbol{\nabla}$ is the momentum operator, and $\boldsymbol{A}$ is the vector potential related to the magnetic field through[‡]

$$\varepsilon_{ij}\partial^i A^j = B,\tag{2.11}$$

whose general solution is

$$A^i = -\frac{B}{2}\left(\varepsilon^{ij}x^j - \partial^i\xi\right),\tag{2.12}$$

where ξ is an arbitrary scalar function that determines the gauge we are working in. It turns out that the allowed energy values for the electron are,

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[§]Strong magnetic field makes the chance of finding the electron with an anti-aligned spin so small that in practice, at least in the first approximation, one can safely ignore the spin freedom of the electron.

[¶]In practice, this can be done, for example, by cooling a sample consisting of an interface of an insulator and a semi-conductor down to almost absolute zero. Page 2 of [Kha05].

[‡]$\varepsilon_{11} = \varepsilon_{22} = 0$, and $\varepsilon_{12} = -\varepsilon_{21} = 1$.

as in the case of harmonic oscillator, evenly spaced and are given by

$$E_n = \hbar\omega_c \left(n + \frac{1}{2} \right), \tag{2.13}$$

where the quantum number n is a non-negative integer and the *cyclotron frequency* ω_c is given by

$$\omega_c = \frac{|e|B}{m_e c}. \tag{2.14}$$

These energy levels are known as *Landau levels* (LL)s in the honor of Lev Landau who, in 1930, solved the problem for the first time. The first energy level $E_0 = 1/2\hbar\omega_c$ is called the *lowest Landau level* (LLL). Although the energy levels are independent of the gauge chosen, in general the form of the corresponding wave functions does depend on the gauge.

In the symmetric gauge where ξ in Equation (2.12) is chosen to be zero, the wave functions corresponding to the nth LL expressed in complex coordinates are given by

$$\psi_{l,n}(z) = \sqrt{\frac{n!}{2\pi 2^l (l+n)!}} \, z^l \, L_n^l \left(\frac{|z|^2}{2l_B^2} \right) \exp\left(-\frac{|z|^2}{4l_B^2} \right). \tag{2.15}$$

In this equation l is an integer not less than $-n$, L_n^l is the associated Laguerre polynomial, $z = x^1 + \mathrm{i}x^2$ where (x^1, x^2) are the Cartesian coordinates of the electron, and

$$l_B = \sqrt{\frac{\hbar c}{|e|B}}. \tag{2.16}$$

The number l_B has the dimension of length and it is called the *magnetic length*, which can be considered as the natural length scale of the system.

In the symmetric gauge, the L^3 component of the angular momentum commutes with Hamiltonian (2.10) and it turns out that for a given value of n the wave function $\psi_{l,n}(z)$ in Equation (2.15) is also an eigenstate of L^3 with eigenvalue $l\hbar$. Note that in complex coordinates

$$L^3 = \hbar(z\partial - \bar{z}\bar{\partial}), \tag{2.17}$$

where $\bar{z}$ is the complex conjugate of z and

$$\partial := \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - \mathrm{i}\partial_2), \tag{2.18}$$

$$\bar{\partial} := \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_1 + \mathrm{i}\partial_2). \tag{2.19}$$

Since l in Equation (2.15) can take any integer value greater or equal than $-n$, each LL is infinitely degenerate. This is a notable characteristic of this problem. The infiniteness of degeneracy stems from the fact that no constraint,

except for that the electron is limited to move in the x^1Ox^2 plane, is imposed on the motion of the electron. However, in practice one always deals with a sample of finite size that confines the electron's motion to a finite region of the x^1Ox^2 plane. Finiteness of the sample, as the following argument shows, provides an upper bound for the degeneracy LLs.

For simplicity we consider only the LLL where the wave functions correspond to zero value for n in Equation (2.15), that is

$$\psi_{l,0}(z) = \frac{1}{\sqrt{2\pi 2^l l!}} z^l \exp\left(-\frac{|z|^2}{4l_B^2}\right), \qquad l = 0, 1, 2, \ldots. \qquad (2.20)$$

By calculating the derivative of $|\psi_{l,0}(z)|^2$ for a given non-negative integer l, it is seen that the maximum of this function occurs at the points of the circle of radius $\sqrt{2l}l_B$ centered at the origin. Hence, for a circular sample of radius R one should not consider the states $\psi_{l,0}(z)$ with $\sqrt{2l}l_B > R$[§] and the degeneracy of LLL is

$$l_{\max} = \frac{R^2}{2l_B^2}. \qquad (2.21)$$

This degeneracy can also be written as

$$l_{\max} = \frac{\pi R^2}{2\pi l_B^2} = \frac{\pi R^2 B}{2\pi l_B^2 B} = \frac{\Phi}{\Phi_0}. \qquad (2.22)$$

Here Φ is the magnetic flux penetrating through the sample and Φ_0 is the *flux quantum* defined by

$$\Phi_0 = \frac{hc}{|e|}, \qquad (2.23)$$

where h is the Planck constant. This ratio is called the *number of flux quanta* and it is denoted by N_Φ ($N_\Phi = l_{\max}$). Another ratio of particular interest in the context of quantum Hall physics is the *filling factor* ν_f. It is defined by

$$\nu_f = \frac{N}{N_\Phi}, \qquad (2.24)$$

where N denotes the number of electrons in the sample. This ratio can be expressed in a different way related to the geometry of the sample. From the discussion above, it is seen that to any value $l\hbar$ ($0 \leqslant l \leqslant l_{\max}$) of the angular momentum one can associate a circle of radius $R_l = \sqrt{2l}l_B$ centered at the origin. The area ΔS encircled by two concentric circles corresponding to two consecutive values l and $l+1$ of the angular momentum is

$$\Delta S = \pi R_{l+1}^2 - \pi R_l^2$$
$$= 2\pi l_B^2, \qquad (2.25)$$

[§]The wave function is zero outside the sample.

hence, from Equations (2.22) and (2.24), one gets

$$\nu_f = \frac{N\Delta S}{S}. \tag{2.26}$$

It turns out that ν_f is equal to ν in Equation (2.8), so from now on we denote it simply by ν.

Now let us look back at the integral and fractional quantum Hall effects. It is clear that the ground state of a FQH system for an integer value of ν is the state corresponding to the case in which all the first ν Landau levels are completely filled[§] and, hence, the ground state is a non-degenerate state. In this case, at least in the first approximation, one can neglect the Coulomb repulsion between electrons since the system is gapped and the typical Coulomb interaction e^2/l_B is much less than this gap $\hbar\omega_c$. Therefore, essentially, IQHE is a *non-interacting* problem and this is why soon after its discovery it was explained theoretically. The many-body wave function is just a single Slater determinant.

As an example, consider the simplest case $\nu = 1$ in which the number of electrons is exactly equal to the number of orbitals in the first LL and let $\Psi_{\nu=1}(z_1, \ldots, z_N)$ denote the unique ground state. This many-body ground state is the following Slater determinant:

$$\Psi_{\nu=1}(z_1, \ldots, z_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{0,0}(z_1) & \cdots & \psi_{0,0}(z_N) \\ \psi_{1,0}(z_1) & \cdots & \psi_{1,0}(z_N) \\ \vdots & & \vdots \\ \psi_{N-1,0}(z_1) & \cdots & \psi_{N-1,0}(z_N) \end{vmatrix}. \tag{2.27}$$

Using Equation (2.20) for the entries of this determinant, we come up with

$$\Psi_{\nu=1}(z_1, \ldots, z_N) = \mathcal{N} \begin{vmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_N \\ \vdots & & \vdots \\ z_1^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} \exp\Big(-\frac{1}{4l_B^2}\sum_{k=1}^{N}|z_k|^2\Big), \tag{2.28}$$

where $\mathcal{N}$ is a constant and the determinant above is the well-known Vandermonde determinant. Using the result of this determinant, one gets

$$\Psi_{\nu=1}(z_1, \ldots, z_N) = \prod_{1\leqslant i<j\leqslant N}(z_i - z_j) \exp\Big(-\frac{1}{4l_B^2}\sum_{k=1}^{N}|z_k|^2\Big), \tag{2.29}$$

up to a normalization constant. Using wave functions (2.15), this example can be generalized to an arbitrary integer filling factor. For $\nu = 2$ case, for example, the reader can refer to [K.J07].

[§]Note that because of Pauli's exclusion principle, no more than one electron can be in the same state.

In contrast, FQHE is a whole new story. The first substantial progress in theoretical explanation of this phenomenon was achieved by Laughlin through his introduction of a set of trial wave functions.

2.3 Laughlin's Wave Function

In a FQH system only a fraction of orbitals in each LL is filled and as mentioned earlier and Figure 2.1 shows, for a FQH system the graph of Hall resistance ρ_H versus the magnetic field B shows plateaus as well, which indicates that the system is gapped. This implies that the Coulomb repulsion between the electrons must definitely be taken into account, since in the absence of the Coulomb interaction any redistribution of electrons within a LL can be done at zero energy cost, giving rise to a large degeneracy. To explain that a gapped quantum Hall state can occur at the observed filling fractions, one needs the Coulomb interaction to lift the degeneracy. This makes a FQH system to be a highly-correlated system and difficult to solve.

In 1983, Robert Laughlin achieved a breakthrough by proposing a set of quantum Hall states in the form of a set of trial wave functions, which were shown to contain the basic features of this phenomenon. Laughlin proposed the ansatz wave function

$$\Psi_m(z_1, \ldots, z_N) = \prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)^m \, \exp\left(-\frac{1}{4l_B^2} \sum_{k=1}^{N} |z_k|^2 \right) \qquad (2.30)$$

to describe the ground state of the FQHE at filling factor $\nu = 1/m$ where m is an odd integer [Lau83]. Laughlin arrived to this wave function by using physical constraints and some intuition. These are outlined in the following.

(i) The suitable wave function should be of the following form[§]

$$\Phi(z_1, \ldots, z_N) = p(z_1, \ldots, z_N) \, \exp\left(-\frac{1}{4l_B^2} \sum_{k=1}^{N} |z_k|^2 \right), \qquad (2.31)$$

where $p(z_1, \ldots, z_N)$ is a polynomial in z_1 till z_N. To write this, Laughlin was inspired by the form of the wave function (2.20) for the LLL states.

(ii) Since this wave function is to describe a system of electrons as fermions, it must be totally anti-symmetric.

(iii) Because of the success of Jastrow-type wave functions in describing the interacting systems with pairwise interactions, as Coulomb interaction in this case, Laughlin assumed the following form

$$p(z_1, \ldots, z_N) = \prod_{1 \leqslant i < j \leqslant N} f(z_i - z_j), \qquad (2.32)$$

[§]One should find out how Φ depends on the filling factor $\nu = 1/m$.

for the $p(z_1, \ldots, z_N)$ polynomial. Here f must be an odd polynomial-function to be consistent with $p(z_1, \ldots, z_N)$ to be totally anti-symmetric.

(iv) Since the total angular momentum along the Ox^3 direction,

$$L^3 = \hbar \sum_{i=1}^{N} (z_i \partial_i - \bar{z}_i \bar{\partial}_i), \tag{2.33}$$

commutes with Coulomb interaction and consequently with the Hamiltonian, Laughlin demanded that the suitable wave function to be an eigenstate of L^3 as well. A simple calculation shows that for this to happen, it is sufficient that the polynomial $p(z_1, \ldots, z_N)$ be an eigenstate of $\hbar \sum_{i=1}^{N} z_i \partial_i$ operator.

It is not hard to see that $f(z) = z^n$, for any odd integer n, is a suitable choice and gives rise to a polynomial $p(z_1, \ldots, z_N)$ that is an eigenstate of $\hbar \sum_{i=1}^{N} z_i \partial_i$ operator. Thus,

$$\Phi(z_1, \ldots, z_N) = \prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)^n \exp\left(- \frac{1}{4l_B^2} \sum_{k=1}^{N} |z_k|^2 \right). \tag{2.34}$$

Now we need to find the appropriate exponent n. This function is supposed to describe interacting electrons in the LLL at filling factor $\nu = 1/m$. The maximum value of angular momentum $l_{\max}$ that one electron in the state (2.34) can have is the maximum power $n(N-1)$ of any one of the variables z_1 till z_N in Φ. Thus according to Equation (2.21), the area S of the sample described by (2.34) is

$$S = 2\pi l_{\max} l_B^2 = 2\pi n(N-1) l_B^2. \tag{2.35}$$

By Equations (2.25) and (2.26), the filling factor ν_f corresponding to the wave function (2.34) is

$$\nu_f = \frac{N}{n(N-1)} = \frac{1}{n}, \tag{2.36}$$

for large values of N and, therefore, $n = m$.

Using his wave function, Laughlin not only explained the $\nu = 1/3$ fractional quantum Hall effect, but also he predicted that quasi-holes with *fractional charge* and *statistics* can exist in FQH systems. The fractional charge of these quasi-holes was observed experimentally in 1997 [dPRH$^+$97, SG97], and Laughlin, Störmer and Tsui were awarded the 1998 Nobel prize in physics.

To understand the quasi-holes, consider a FQH system in the $x^1 O x^2$ plane subjected to a magnetic field in the positive Ox^3 direction that exhibits the fractional value $\nu = 1/m$. Suppose that its state is modeled by the Laughlin state Ψ_m given by Equation (2.30). Following Laughlin, we locally increase the magnetic field at the origin by one flux quantum Φ_0. This can be thought

to be done by considering an infinitesimally thin and infinitely long solenoid threading normally into the system at the origin and varying slowly (adiabatically) the current through it from zero to some appropriate value and in an appropriate direction. Such solenoid is referred to as a "flux tube." Variation of the magnetic field at the origin generates an electric field $\boldsymbol{E}$ curling around the origin in a direction resisting this change. This electric field in turn generates an electric current of density $\boldsymbol{j}$ that relates itself to the Hall conductivity tensor σ and the electric field $\boldsymbol{E}$ through the following equation:

$$\boldsymbol{j} = \sigma \boldsymbol{E}. \tag{2.37}$$

According to Equation (2.9), the entries of the main diagonal of the conductivity tensor are zero for a FQH system on a plateau and, therefore, $\boldsymbol{j}$ lies along the radial direction towards the origin, where the flux tube is located. Thus, Equation (2.37) reduces to the following one

$$j_r = \sigma_H E_\phi, \tag{2.38}$$

with j_r the radial component of current density, E_ϕ the azimuthal component of the electric field, and σ_H is as given in Equation (2.9). This current density indicates that the electrons flows out from a small region confined by a small circle centered at the origin, where the flux tube is located, and making a "hole" behind them known as *quasi-holes*. During this adiabatic process, the ground state Ψ_m evolves to the ground state of the final Hamiltonian where the magnetic flux is now increased by one flux quantum Φ_0. This excess of magnetic flux can be gauged away and we are left with the new exact quasi-hole ground state of the Hamiltonian. It turns out that this small region can act as a particle on its own. Laughlin proposed the following trial wave function

$$\Psi_m^{\text{q.h.}}(z_1, \ldots, z_N) = \Psi_m(z_1, \ldots, z_N) \prod_{1 \leqslant k \leqslant N} z_k \tag{2.39}$$

for theoretical explanation of a FQH system with one quasi-hole at the origin. Though not a ground state for the Coulomb interaction, $\Psi_m^{\text{q.h.}}(z_1, \ldots, z_N)$ is exact for the model Hamiltonian. In general, if the magnetic flux is slowly changed from zero up to one flux quantum Φ_0 at n local points with complex coordinates w_1 till w_n, Equation (2.39) then takes the following form:

$$\Psi_m^{\text{q.h.}}(z_1, \ldots, z_N) = \Psi_m(z_1, \ldots, z_N) \prod_{\substack{1 \leqslant k \leqslant N \\ 1 \leqslant l \leqslant n}} (z_k - w_l). \tag{2.40}$$

Wave functions $\Psi_m^{\text{q.h.}}(z_1, \ldots, z_N)$ are known as quasi-hole *excitations* of Laughlin states $\Psi_m(z_1, \ldots, z_N)$.

Let us go back to the simple case of one quasi-hole at the origin to explore some interesting features of quasi-holes. The state (2.39) has a lack of charge

of some magnitude Q at the origin. By Faraday's law

$$\oint_\Gamma \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = -\frac{1}{c}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{2.41}$$

where Γ is a small circle of radius R centered at the origin. This gives rise to

$$E_\phi = -\frac{1}{2\pi Rc}\frac{\mathrm{d}\Phi}{\mathrm{d}t}, \tag{2.42}$$

and

$$Q = \int 2\pi R|j_r|\,\mathrm{d}t = \frac{1}{c}\sigma_H\Phi_0$$
$$= \nu|e|. \tag{2.43}$$

This means that this excitation can be regarded as the one with a *fractional* charge of magnitude $\nu|e|$.

Fractional statistics is another amazing property of quasi-hole particles. From quantum mechanics we know that a particle in three spatial dimensions can be either a boson or a fermion. We also know that if two bosons are exchanged the wave function is not affected

$$\psi(z_1, z_2) = \mathrm{e}^{\mathrm{i}0\pi}\psi(z_2, z_1), \tag{2.44}$$

but if two fermions are exchanged, the wave function picks up a minus sign,

$$\psi(z_1, z_2) = \mathrm{e}^{\mathrm{i}1\pi}\psi(z_2, z_1). \tag{2.45}$$

These together with the Pauli exclusion principle constitutes the content of the *spin statistics* theorem. Here 0 and 1 in the equations above can, of course, be replaced by any even and odd *integers*, respectively. Therefore, bosons and fermions are said to have integer statistics. In a FQH system the quasi-holes obey fractional statistics meaning that if two of them are slowly interchanged, the wave function undergoes as follows

$$\psi(w_1, w_2) = \mathrm{e}^{\mathrm{i}\alpha\pi}\psi(w_2, w_1), \tag{2.46}$$

where α is a *fraction* strictly between zero and one and it turns out that α is the same as the fraction ν, as was shown by Arovas, Schrieffer and Wilczek [ASW84], by calculating the phase associated with process of adiabatically exchanging two quasi-holes.

Reduced Density Operator and Schmidt Decomposition

As mentioned in Chapter 1, the goal of this thesis is to determine the rank of the reduced density operator corresponding to a quantum Hall system subjected to a bipartition of the system known as the particle cut. This chapter provides a short reminder of the notion of the reduced density operator and an operation called partial trace. It also recalls the well-known Schmidt Theorem from linear algebra and provides a theorem to enable us to determine the rank of the reduced density operator of a pure state that is decomposed into a single sum composed of products of linearly independent functions, instead of orthonormal functions, as in the case of Schmidt Theorem. The material in this chapter, except for the last theorem, can be found in [NC10] in more detail.

3.1 Reduced Density Operator

Consider a composite system $\mathcal{S}$ that is composed of two subsystems A and B. If $\mathcal{H}$, $\mathcal{H}^A$, and $\mathcal{H}^B$ are Hilbert spaces corresponding to systems $\mathcal{S}$, A, and B, respectively, then from quantum mechanics one knows that $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B$. In this chapter, it is assumed that $\mathcal{H}^A$ and $\mathcal{H}^B$, and consequently, $\mathcal{H}$ are finite-dimensional Hilbert spaces.

Let the system $\mathcal{S}$ be described by a density operator ρ^{AB}. The *reduced density operator* ρ^A of subsystem A is defined by

$$\rho^A = \mathrm{tr}_B(\rho^{AB}), \tag{3.1}$$

where tr_B is a linear map, called the *partial trace* over system B, that assigns to each linear operator on $\mathcal{H}^A$ a linear operator on $\mathcal{H}$ according to the following rule

$$\mathrm{tr}_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2|\,\mathrm{tr}(|b_1\rangle\langle b_2|), \tag{3.2}$$

where $|a_1\rangle$ and $|a_2\rangle$ are two states in $\mathcal{H}^A$ and $|b_1\rangle$ and $|b_2\rangle$ are two states in $\mathcal{H}^B$. Note that Equation (3.2) together with the linearity of tr_B suffices to

know how tr_B acts on a generic Hermitian operator $A \otimes B$ on $\mathcal{H}^A \otimes \mathcal{H}^B$. If $\mathcal{S}$ is in the pure state $|\Psi\rangle$, then Equation (3.1) reduces to the simple form

$$\rho^A = \mathrm{tr}_B(|\Psi\rangle\langle\Psi|). \tag{3.3}$$

The reduced density operator ρ^B of subsystem B is defined similarly. For further studies the reader is referred to [NC10].

3.2   Schmidt Decomposition

Consider a composite system $\mathcal{S}$ composed of two parts A and B. We know that if $\big\{\, |u_i\rangle \mid 1 \leqslant i \leqslant N_A \,\big\}$ is an orthonormal basis for the N_A-dimensional state space $\mathcal{H}^A$ of subsystem A and $\big\{\, |v_i\rangle \mid 1 \leqslant i \leqslant N_B \,\big\}$ is an orthonormal basis for the N_B-dimensional state space $\mathcal{H}^B$ of subsystem B, then the set

$$\big\{\, |u_i\rangle \otimes |v_j\rangle \mid 1 \leqslant i \leqslant N_A \,,\, 1 \leqslant j \leqslant N_B \,\big\} \tag{3.4}$$

is an orthonormal basis for the $(N_A N_B)$-dimensional state space $\mathcal{H}^A \otimes \mathcal{H}^B$ of the whole system $\mathcal{S}$. Therefore, if $|\Psi\rangle$ is a normalized pure state of the system $\mathcal{S}$, then

$$|\Psi\rangle = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} c_{ij} \, u_i \otimes v_j, \tag{3.5}$$

for some complex numbers c_{ij} with $\sum_{i=1}^{N_A} \sum_{j=1}^{N_B} |c_{ij}|^2 = 1$. That is any pure state of the whole system $\mathcal{S}$ can be written as a double sum as is indicated in Equation (3.5). The following theorem, known as the *Schmidt Decomposition* Theorem, asserts that it is always possible to write $|\Psi\rangle$ as a single sum for appropriately chosen orthonormal subsets of corresponding state spaces. The interested reader can refer to [NC10] for a proof.

Theorem 3.1 (Schmidt Decomposition). *Let $|\Psi\rangle$ be a normalized pure state of a composite system $\mathcal{S}$ composed of subsystems A and B with corresponding Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$ of dimensions N_A and N_B, respectively. There exist an orthonormal set of states $\big\{\, |\phi_i^A\rangle \,\big\}_i$ in $\mathcal{H}^A$ and an orthonormal set of states $\big\{\, |\phi_i^B\rangle \,\big\}_i$ in $\mathcal{H}^B$ such that*

$$|\Psi\rangle = \sum_{i=1}^{r} \lambda_i \, |\phi_i^A\rangle \otimes |\phi_i^B\rangle, \tag{3.6}$$

where $r = \min\{N_A, N_B\}$ and λ_i's are non-negative real numbers such that $\sum_{i=1}^{r} \lambda_i^2 = 1$.

The number of strictly positive λ_i's in the theorem above is called the *Schmidt number* of $|\Psi\rangle$.

Corollary 3.2. *As in the theorem above, let system $\mathcal{S}$ be in the pure state $|\Psi\rangle$ given by Equation (3.6). The rank of the reduced density operator of the subsystem with Hilbert space of lower dimension is equal to the Schmidt number of $|\Psi\rangle$*

Proof. Without loss of generality let $N_A \leqslant N_B$. Then plugging Equation (3.6) into Equation (3.3) and making use of Equation (3.2) gives

$$\rho^A = \sum_{i=1}^{N_A} \sum_{j=1}^{N_A} \lambda_i \lambda_j \, |\phi_i^A\rangle\langle\phi_j^A| \, \mathrm{tr}(|\phi_i^B\rangle\langle\phi_j^B|). \tag{3.7}$$

On the other hand

$$\mathrm{tr}(|\phi_i^B\rangle\langle\phi_j^B|) = \langle\phi_j^B|\phi_i^B\rangle = \delta_{ij}. \tag{3.8}$$

Hence,

$$\rho^A = \sum_{i=1}^{N_A} \lambda_i^2 \, |\phi_i^A\rangle\langle\phi_i^A|. \tag{3.9}$$

Thus, Spectral Decomposition Theorem implies that the spectrum of ρ^A is the set $\left\{\, \lambda_i^2 \mid 1 \leqslant i \leqslant N_A \,\right\}$, and therefore, the rank of ρ^A is the number of non-zero elements in this set, which is the Schmidt number of $|\Psi\rangle$. $\qquad\square$

Schmidt's Theorem motivates Theorem 3.3. This theorem asserts that if a pure state $|\Psi\rangle$ is decomposed as in Equation (3.6), one can still conclude that the number r is the Schmidt number of $|\Psi\rangle$, even if, instead of being orthonormal, the states $\{|\phi_i^A\rangle\}_i$ and $\{|\phi_i^B\rangle\}_i$ are only known to be linearly independent in their corresponding Hilbert spaces. This theorem would be helpful in Section 5.1 where we need to determine the rank of the reduced density operator of the Laughlin state.

Theorem 3.3. *Let $|\Psi\rangle$ be a normalized pure state of a composite system $\mathcal{S}$ composed of subsystems A and B with corresponding Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$ of dimensions N_A and N_B, respectively. If*

$$|\Psi\rangle = \sum_{i=1}^{r} \xi_i \, |\varphi_i^A\rangle \otimes |\varphi_i^B\rangle, \tag{3.10}$$

where $r \leqslant \min\{N_A, N_B\}$, ξ_i's are non-zero numbers, and $\left\{\, |\varphi_i^A\rangle \mid 1 \leqslant i \leqslant r \,\right\}$ and $\left\{\, |\varphi_i^B\rangle \mid 1 \leqslant i \leqslant r \,\right\}$ are linearly independent subsets of $\mathcal{H}^A$ and $\mathcal{H}^B$, respectively, then the rank of the reduced density operators ρ^A and ρ^B is equal to r.

Proof. In a similar way that gave rise to Equation (3.7), one gets

$$\rho^A = \sum_{i=1}^{r} \sum_{j=1}^{r} \bar{\xi}_j \xi_i \, \langle\varphi_j^B|\varphi_i^B\rangle |\varphi_i^A\rangle\langle\varphi_j^A|. \tag{3.11}$$

Linearly independent subsets $\left\{ \, |\varphi_i^A\rangle \mid 1 \leqslant i \leqslant r \, \right\}$ and $\left\{ \, |\varphi_i^B\rangle \mid 1 \leqslant i \leqslant r \, \right\}$ can be extended to form bases $\left\{ \, |\varphi_i^A\rangle \mid 1 \leqslant i \leqslant N_A \, \right\}$ and $\left\{ \, |\varphi_i^B\rangle \mid 1 \leqslant i \leqslant N_B \right\}$ for $\mathcal{H}^A$ and $\mathcal{H}^B$, respectively. Let $\left\{ \, |\alpha_i^A\rangle \mid 1 \leqslant i \leqslant N_A \right\}$ and $\left\{ \, |\alpha_i^B\rangle \mid 1 \leqslant i \leqslant N_B \right\}$ be dual bases of the bases mentioned above, thus

$$\langle \alpha_i^A | \varphi_j^A \rangle = \delta_{ij}, \tag{3.12}$$

$$\langle \alpha_i^B | \varphi_j^B \rangle = \delta_{ij}. \tag{3.13}$$

The goal is to show that the kernel of ρ^A is of dimension $N_A - r$ that implies the rank of ρ^A is r. Let $|\Phi^A\rangle$ be a vector in $\mathcal{H}^A$ and

$$|\Phi^A\rangle = \sum_{k=1}^{N_A} \alpha_k^A |\alpha_k^A\rangle, \tag{3.14}$$

for numbers α_k^A. It is seen that

$$\rho^A |\Phi^A\rangle = \sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_j^A \bar{\xi}_j \xi_i \, \langle \varphi_j^B | \varphi_i^B \rangle |\varphi_i^A\rangle. \tag{3.15}$$

Therefore, if α_j^A's are zero for $1 \leqslant j \leqslant r$, then $|\Phi^A\rangle$ is in the kernel of ρ^A. The converse is also true. To prove it, assume that $|\Phi^A\rangle$ is in the kernel of ρ^A, then since $|\varphi_i^A\rangle$'s are linearly independent, from Equation (3.15) we get

$$\left(\sum_{j=1}^{r} \alpha_j^A \bar{\xi}_j \, \langle \varphi_j^B | \right) (\xi_i \, |\varphi_i^B\rangle) = 0, \tag{3.16}$$

for all $1 \leqslant i \leqslant r$. Since ξ_i's are non-zero, the left factor above is orthogonal to the subspace spanned by $\left\{ \, |\varphi_i^B\rangle \mid 1 \leqslant i \leqslant r \, \right\}$. This factor is also orthogonal to the complement subspace spanned by $\left\{ \, |\varphi_i^B\rangle \mid r+1 \leqslant i \leqslant N_B \, \right\}$, so it should vanish. Since $|\varphi_j^B\rangle$'s are linearly independent, this implies that $\alpha_j^A = 0$ for $1 \leqslant j \leqslant r$. The proof for ρ^B is similar. $\qquad \square$

Any decomposition as the one in Equation (3.10) is referred to as a *weak Schmidt decomposition* of $|\Psi\rangle$.

Symmetric Functions

This chapter touches only a few aspects of the rich theory of symmetric functions to that extent that is necessary to understand the proof of Theorem 4.4. This theorem provides a decomposition for the double product $\prod_{i=1}^{n}\prod_{j=1}^{m}(1 + x_iy_j)$ in n variables x_1 till x_n and m variables y_1 till y_m in the form of a single sum of terms with each term being a product of symmetric polynomials, one of which a polynomial only in x_i's and the other one a polynomial only in y_j's. This, in turn, makes it possible to find a candidate for a weak Schmidt decomposition of a generic Laughlin's wave function at an arbitrary filling factor $\nu = 1/m$, which is the content of Chapter 5. The material in this chapter is a standard one on the theory of symmetric functions [Sta99, Mac95] and the theory of partitions [And84].

4.1 General Terminology

As it becomes clear while we proceed, one needs to introduce different bases for the vector space of symmetric functions with rational coefficients and in the meanwhile one also needs to be able to multiply two elements of this vector space. The former is possible in a vector space structure but the latter is not. This motivates the introduction of an algebra. The fundamental mathematical object in this chapter is the *algebra of symmetric functions* over the *field* of rational numbers $\mathbb{Q}$. First, we the describe words written in italics.

For the purpose of this thesis, it is sufficient to think of the field of rational numbers as the ordinary set of rational numbers together with addition and multiplication of rational numbers.

A vector space $\mathcal{A}$ over $\mathbb{Q}$ equipped with an extra operation

$$\begin{cases} \times : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A} \\ \times(u, v) = u \times v \end{cases},$$

called *vector product*, is said to be an *algebra* over $\mathbb{Q}$ if for any rational numbers a and b, and any vectors u, v, and w in $\mathcal{A}$,

$$u \times (av + bw) = a(u \times v) + b(u \times w), \tag{4.1}$$

$$(av + bw) \times u = a(v \times u) + b(w \times u). \tag{4.2}$$

Usually one drops $\times$ and simply writes uv for $u \times v$. Moreover, if for any u and v in $\mathcal{A}$

$$uv = vu, \tag{4.3}$$

$\mathcal{A}$ is said to be *commutative*, if for any u, v, and w in $\mathcal{A}$

$$u(vw) = (uv)w, \tag{4.4}$$

$\mathcal{A}$ is said to be *associative*, and if there exists an element i in $\mathcal{A}$ such that

$$ui = iu, \tag{4.5}$$

for any u in $\mathcal{A}$, then $\mathcal{A}$ is said to be *unital* and i, which is easily seen to be unique, is called the *unit* vector of $\mathcal{A}$. In this thesis, the zero vector is denoted by o.

Let us emphasis again that an algebra, a priori, is a vector space. Therefore, any notion that is applicable to a vector space such as the notion of a linear combination of vectors, linear independence, basis, etc., is also applicable to an algebra.

One can also define a subalgebra. Let $\mathcal{A}$ be an algebra over $\mathbb{Q}$. A vector subspace of $\mathcal{A}$ is said to be a *subalgebra* of $\mathcal{A}$ if it is closed with respect to the vector product. Another terminology in this context is the notion of an ideal. A vector subspace $\mathcal{I}$ of an algebra $\mathcal{A}$ is said to be a *left ideal* (*right ideal*) of $\mathcal{A}$ if for any u in $\mathcal{A}$ and any ι in $\mathcal{I}$,

$$u\iota \in \mathcal{I} \ (\iota u \in \mathcal{I}), \tag{4.6}$$

and it is said to be an *ideal* of $\mathcal{A}$ if both statements in (4.6) are satisfied.

As an example, the vector space

$$\mathcal{A} = \{(a, b, c) \mid a, b, c \in \mathbb{Q}\}, \tag{4.7}$$

with vector addition and scalar multiplication defined component-wise, together with the ordinary cross product of vectors as the vector product, forms an algebra over $\mathbb{Q}$ which is neither commutative nor associative. Here $(0, 0, 0)$ is the zero vector but there is no unit vector. Sets $\{(0, 0, 0)\}$ and $\mathcal{A}$ are the only ideals of this algebra.

Another example is the algebra of all polynomials in n variables x_1 till x_n with rational coefficients, which is denoted by $\mathbb{Q}[x_1, \ldots, x_n]$. Here the vector addition and scalar multiplication are just the standard addition of polynomials and multiplication of a polynomial by a rational number, the zero vector is the zero polynomial, and the vector product is the ordinary multiplication of polynomials. This is a commutative, associative, and unital algebra and the polynomial i such that $i(x_1, \ldots, x_n) = 1$ is the unit vector. Furthermore, from the properties of polynomials we know that this algebra does not contain a *zero divisor*. An algebra $\mathcal{A}$ is said to be without zero

divisor if uv is non-zero for any two non-zero vectors u and v in $\mathcal{A}$. The subset consisting of symmetric polynomials, as are defined shortly, of $\mathbb{Q}[x_1, \ldots, x_n]$ is a subalgebra and it is denoted by $\Lambda[x_1, \ldots, x_n]$ or, when the variables are not important or they are clear from the context, briefly by Λ_n. As becomes clear after the definition of a symmetric polynomial, Λ_n is not an ideal of $\mathbb{Q}[x_1, \ldots, x_n]$.

A polynomial $s(x_1, \ldots, x_n)$ in $\mathbb{Q}[x_1, \ldots, x_n]$ is said to be *symmetric* if

$$s\big(x_{\sigma(1)}, \ldots, x_{\sigma(n)}\big) = s(x_1, \ldots, x_n), \tag{4.8}$$

for any permutation σ of the set $\{1, 2, \ldots, n\}$. The polynomial

$$s(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 + x_3 x_1 - \frac{3}{2} x_1 x_2 x_3, \tag{4.9}$$

is a symmetric polynomial in $\mathbb{Q}[x_1, x_2, x_3]$. It is clear that any constant polynomial, including the zero and the unit polynomials, is symmetric and, therefore, Λ_n is a unital (sub)algebra. Obviously, any polynomial in just one variable is symmetric.

A polynomial $h(x_1, \ldots, x_n)$ is called a *homogeneous* polynomial of *total degree D* if

$$h(tx_1, \ldots, tx_n) = t^D h(x_1, \ldots, x_n), \tag{4.10}$$

for any number t. The polynomial

$$h(x_1, x_2, x_3) = 2x_1 x_2 x_3 + \frac{1}{2} x_1^2 x_3 \tag{4.11}$$

is a homogeneous polynomial of total degree three that is not symmetric and

$$f(x_1, x_2, x_3) = x_1 x_2 x_3 - \frac{1}{2}(x_1^2 x_2 + x_1 x_2^2 + x_2^2 x_3 + x_2 x_3^2 + x_3^2 x_1 + x_1 x_3^2) \tag{4.12}$$

is a homogenous symmetric polynomial of total degree three as well. In this thesis, the greatest exponent of any one of the variables in a homogenous symmetric polynomial is referred to as the *degree* of the polynomial[§]. For example, although the polynomial (4.12) is of total degree three, it is of degree two. Any non-zero constant polynomial is a homogeneous polynomial of both total degree and degree equal to zero. Of course, the zero polynomial is also a homogeneous polynomial but of undefined degree and total degree.

In this context, the subset of $\mathbb{Q}[x_1, \ldots, x_n]$ consisting of homogeneous symmetric polynomials of *total* degree D together with the zero polynomial is denoted by $\Lambda_{n,D}$ and the subset of $\mathbb{Q}[x_1, \ldots, x_n]$ consisting of those polynomials of *degree* at most d together with the zero polynomial is denoted by Λ_n^d. Although $\Lambda_{n,D}$ and Λ_n^d are vector subspaces of $\mathbb{Q}[x_1, \ldots, x_n]$, they do not form a subalgebra since none of them is closed under the multiplication of polynomials.

[§]This is not the same as in the mathematics literatures.

In Section 4.3, different bases for Λ_n, $\Lambda_{n,D}$, and Λ_n^d are introduced. The following observation is helpful in this regard. Any symmetric polynomial in n variables can be uniquely decomposed into a sum of homogeneous polynomials of different total degrees in the same variables. This implies that

$$\Lambda_n = \bigoplus_{D \geqslant 0} \Lambda_{n,D}, \tag{4.13}$$

where $\oplus$ is the direct sum of vector subspaces. Hence, the union of the bases for subspaces $\Lambda_{n,D}$ constitutes a basis for Λ_n.

Another concept that needs to be defined is the concept of a generating set. A set

$$G = \big\{ \, s_1(x_1, \ldots, x_n), \ldots, s_m(x_1, \ldots, x_n) \, \big\} \tag{4.14}$$

of symmetric polynomials is said to be a *generating set* for $\Lambda_n[x_1, \ldots, x_n]$ if any polynomial in $\Lambda_n[x_1, \ldots, x_n]$ can be written as a polynomial with rational coefficients in elements of G. In other words, if for every polynomial $s(x_1, \ldots, x_n)$ in $\Lambda_n[x_1, \ldots, x_n]$ there exists a polynomial $r(X_1, \ldots, X_m)$ in $\mathbb{Q}[X_1, \ldots, X_m]$ such that

$$r\big(s_1(x_1, \ldots, x_n), \ldots, s_m(x_1, \ldots, x_n)\big) = s(x_1, \ldots, x_n). \tag{4.15}$$

The polynomial r is called a *generating polynomial* for s and Equation (4.15) is read as that s is generated by s_1 till s_m through r.

The set G above is said to be *algebraically independent* if the zero polynomial in n variables, $o(x_1, \ldots, x_n) \equiv 0$, can be generated by s_1 till s_m only through the zero polynomial in m variables, $o(X_1, \ldots, X_m) \equiv 0$. In other words, G is said to be algebraically independent if

$$r\big(s_1(x_1, \ldots, x_n), \ldots, s_m(x_1, \ldots, x_n)\big) = 0, \tag{4.16}$$

implies $r = o$. One should note that algebraic independence and linear independence are not the same. Of course, every algebraically independent set of polynomials is also a linearly independent set. For example the polynomials $p_1(x_1, x_2) = x_1 + x_2$, $p_2(x_1, x_2) = x_1^2 + x_2^2$, and $p_3(x_1, x_2) = x_1^3 + x_2^3$ are linearly independent but they are not algebraically independent. They are linearly independent since for rational numbers a, b, and c the statement

$$\forall x_1, x_2 \; : \; a\,p_1 + b\,p_2 + c\,p_3 = o \tag{4.17}$$

implies that $a = b = c = 0$. They are not algebraically independent since there is a *non-zero* polynomial

$$r(X_1, X_2, X_3) = \frac{1}{2}X_1^3 - \frac{3}{2}X_1 X_2 + X_3 \tag{4.18}$$

such that

$$r\big(p_1(x_1, x_2), p_2(x_1, x_2), p_3(x_1, x_2)\big) = 0. \tag{4.19}$$

In the theory of symmetric functions it is almost always easier and sometimes mandatory to work with infinite number of variables. Then, if necessary, one can draw conclusions for the finite-number case by letting all but finitely many of the variables go to zero. In this context, one writes $\Lambda[x_1, x_2, x_3 \ldots]$, or Λ for short, to denote the algebra of all symmetric functions[§] in x_1, x_2, x_3, $\ldots$. The notion of a symmetric polynomial extends itself naturally to that of a symmetric function. A symmetric function $s(x_1, x_2, x_3, \ldots)$ can be viewed as a polynomial that does not change under the exchange of any pair of its variables. A homogeneous function can also be defined similar to a homogeneous polynomial. In this thesis $\Lambda_{,D}$ and Λ^d denote the sets consisting of homogeneous symmetric functions of total degree D and degree of at most d, respectively, in an infinite number of variables. In this case, the analogue of Equation (4.13) is

$$\Lambda = \bigoplus_{D \geqslant 0} \Lambda_{,D}. \tag{4.20}$$

The notion of a generating set can also be generalized to the case of infinite number of variables in a natural way.

Finally, a map ψ of an algebra $\mathcal{A}$ into itself is called an *algebra endomorphism* on $\mathcal{A}$, if for any two elements u and v in $\mathcal{A}$, and any two numbers a and b in $\mathbb{Q}$,

$$\psi(au + bv) = a\psi(u) + b\psi(v), \tag{4.21}$$

$$\psi(uv) = \psi(u)\psi(v). \tag{4.22}$$

That is, if ψ preserves the linear combination of vectors as well as the vector product. By its definition, it is readily seen that an algebra endomorphism ψ on an algebra $\mathcal{A}$ is uniquely determined if one knows how ψ acts on the elements of an algebraically independent generating set for $\mathcal{A}$. Furthermore, if ψ is an algebra endomorphism on a unital algebra $\mathcal{A}$ with unit vector i and without a zero divisor, then $\psi(i) = 0$ or $\psi(i) = i$.

4.2 Partitions of Non-negative Integers

In order to be able to delve further into the theory of symmetric functions, one needs some acquaintance with the notion of partitions of non-negative integers.

4.2.1 Definitions and Notations

Let n be a positive integer. An infinite sequence

$$\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r, 0, 0, 0 \ldots), \tag{4.23}$$

[§]When the number of variables is infinite, we talk about symmetric functions rather than symmetric polynomials.

consisting of positive integers λ_1 till λ_r together with infinite number of zeros at the end, is said to be an *r-partition* of n, if

$$\lambda_1 \geqslant \cdots \geqslant \lambda_r, \tag{4.24}$$

and

$$\sum_{i=1}^{r} \lambda_i = n. \tag{4.25}$$

For simplicity in writing, one usually suppresses the infinite number of tail-zeros. Thus, the partition (4.23) is simply written as $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$. For example, $(3, 1, 1, 0, 0, \ldots)$ is a partition of 5 that is usually written as $(3, 1, 1)$. We agree that the infinite sequence $(0, 0, 0, \ldots)$, or the empty sequence $()$ after suppressing tail-zeros, is the only partition of zero. This partition is called the *empty* partition and it is denoted by $\emptyset$. In this thesis, although not a standard one, a partition is denoted by a bold Greek letter and its parts are denoted by the same letter in its ordinary style subscripted by positive integers.

To show that $\boldsymbol{\lambda}$ is a partition of n, one writes $\boldsymbol{\lambda} \vdash n$. The number n is referred to as the *weight* of $\boldsymbol{\lambda}$ and it is denoted by $|\boldsymbol{\lambda}|$, so $|\emptyset| = 0$. Each (non-zero) λ_i is called a *part* of $\boldsymbol{\lambda}$. The number of parts of $\boldsymbol{\lambda}$ is defined to be its *length* and it is denoted by $l(\boldsymbol{\lambda})$, so $l(\emptyset) = 0$. The set of all partitions of n is denoted by $\text{Par}(n)$. For instance,

$$\text{Par}(5) = \{(5), (4, 1), (3, 2), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)\}. \tag{4.26}$$

The set of all partitions is denoted by $\mathscr{P}$ and it is defined by

$$\mathscr{P} = \bigcup_{n \geqslant 0} \text{Par}(n). \tag{4.27}$$

The *multiplicity* of a positive integer i in a partition $\boldsymbol{\lambda}$ is denoted by $m_i(\boldsymbol{\lambda})$ or, when the partition $\boldsymbol{\lambda}$ is clear from the context, briefly by m_i and it is defined to be the the number of parts of $\boldsymbol{\lambda}$ that are equal to i. In other words,

$$m_i(\boldsymbol{\lambda}) = \text{Card}\{ j \mid \lambda_j = i \}, \tag{4.28}$$

where "Card" refers to cardinality or the number of elements of a set. This provides still another useful notation for a partition. Using the notion of multiplicity any partition $\boldsymbol{\lambda}$ can be written as

$$\boldsymbol{\lambda} = (1^{m_1} 2^{m_2} 3^{m_3} \cdots), \tag{4.29}$$

where i^{m_i}, for any $i \geqslant 1$, means that there are exactly m_i parts in $\boldsymbol{\lambda}$ that are equal to i. In this notation

$$\sum_{i \geqslant 1} i m_i = |\boldsymbol{\lambda}|. \tag{4.30}$$

For example, $\boldsymbol{\lambda} = (1^2 2^0 3^4 4^0 5^0 \cdots)$ is the partition $(3, 3, 3, 3, 1, 1)$ of 14 that is usually written briefly as $\boldsymbol{\lambda} = (1^2 3^4)$.

4.2.2 Graphical Representation of Partitions

To make working with partitions easier, one can associate a graphical representation with a partition in some way. One of these representations is called the *Ferrers* diagram. By definition, the Ferrers diagram associated with a non-empty partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ is the set of all points $(i, j) \in \mathbb{N}^2$ such that $1 \leqslant j \leqslant \lambda_i$. Here $\mathbb{N}$ refers to the set of all positive integers. In this thesis matrix convention is adopted to draw the Ferrers diagrams, namely, the row index i increases as one goes downwards and the column index j increases as one goes from left to right. Figure 4.1 shows the Ferrers diagram for $\boldsymbol{\lambda} = (5, 3, 3, 2, 1, 1)$ as a partition of 15.

$$
\begin{matrix}
\bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & & \\
\bullet & \bullet & & & \\
\bullet & & & & \\
\bullet & & & &
\end{matrix}
$$
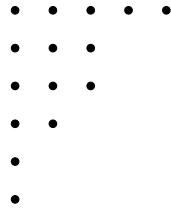
Figure 4.1: Ferrers diagram for $\boldsymbol{\lambda} = (5, 3, 3, 2, 1, 1)$.

Given a partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$, its *conjugate* $\boldsymbol{\lambda}' = (\lambda_1', \ldots, \lambda_s')$ is defined by

$$
\lambda_i' = \mathrm{Card}\{\, j \mid \lambda_j \geqslant i \,\} \tag{4.31}
$$

for any i, $i = 1, 2, \ldots, s$. If $\boldsymbol{\lambda}$ is a partition of a non-negative integer n, then it is readily seen that $\boldsymbol{\lambda}'$ is also a partition of n. Simply, $\boldsymbol{\lambda}'$ can be considered as a partition whose Ferrer diagram is the transpose of the Ferrer diagram of $\boldsymbol{\lambda}$. Transpose of a diagram means a diagram obtained by reflection in the main diagonal. For partition $\boldsymbol{\lambda}$ in Figure 4.1, the conjugate is $\boldsymbol{\lambda}' = (6, 4, 3, 1, 1)$. This is shown schematically in Figure 4.2.

$$
\begin{matrix}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet & & \\
\bullet & \bullet & \bullet & & & \\
\bullet & & & & & \\
\bullet & & & & &
\end{matrix}
$$
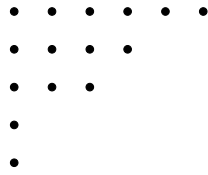
Figure 4.2: Ferrers diagram for $\boldsymbol{\lambda}' = (6, 4, 3, 1, 1)$.

It is obvious that $\emptyset' = \emptyset$, $(\boldsymbol{\lambda}')' = \boldsymbol{\lambda}$, $\lambda_1' = l(\boldsymbol{\lambda})$, $\lambda_1 = l(\boldsymbol{\lambda}')$, and also

$$
m_i(\boldsymbol{\lambda}) = \lambda_i' - \lambda_{i+1}'. \tag{4.32}
$$

The idea of representing partitions by some pictorial image provides us with a strong tool that enables us to conclude simple but non-trivial results about partitions of integers. For instance, using Ferrers diagrams, one can

easily convince oneself that the number of partitions of a positive integer n with at most m parts is equal to the number of partitions of n in which no part exceeds m. To see this, one just needs to establish a one-to-one correspondence between the two classes of partitions of n by mapping each partition from one class onto its conjugate, which is definitely an element from the other class, since from the graphical representation introduced above it is obvious that under conjugation a partition of n with at most m parts is mapped onto a partition of n in which no part is greater than m and vice versa.

4.2.3 Orders on $\mathrm{Par}(n)$

Later on, it becomes important to be able to arrange the partitions of an integer with respect to some kind of order. First, let us have a short review on the definition of a relation on a set and the notions of partial and total orders on that set.

A subset $\mathcal{R}$ of the Cartesian product $A \times A$ of a set A by itself is called a *relation on A*. It is said that $\mathcal{R}$ is a *partial order* on A, or A is *partially ordered by $\mathcal{R}$* if $\mathcal{R}$ is

(i) *reflexive*, that is, for any x in A, (x, x) is in $\mathcal{R}$,

(ii) *anti-symmetric*, that is, for any x and y in A, if (x, y) and (y, x) are both in $\mathcal{R}$ then $x = y$,

(iii) *transitive*, that is, for any x, y, and z in A, if (x, y) and (y, z) are both in $\mathcal{R}$ then (x, z) is in $\mathcal{R}$ as well.

If $\mathcal{R}$ is a relation on A, one usually writes $x\mathcal{R}y$ instead of $(x, y) \in \mathcal{R}$ and reads it as "*x is $\mathcal{R}$-related to y*."

Let n be a non-negative integer. It is straight forward to check the properties (i)–(iii) above for the relation $\preceq$ defined on $\mathrm{Par}(n)$ by

$$\forall \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathrm{Par}(n),\ \boldsymbol{\lambda} \preceq \boldsymbol{\mu} \longleftrightarrow \forall i \geqslant 1 : \lambda_1 + \cdots + \lambda_i \leqslant \mu_1 + \cdots + \mu_i, \quad (4.33)$$

and observe that it defines a partial order on $\mathrm{Par}(n)$. This relation is called the *dominance* or *natural* order on $\mathrm{Par}(n)$. For example, partitions of the set $\mathrm{Par}(5)$ in (4.26), are ordered as follows:

$$(1, 1, 1, 1, 1) \preceq (2, 1, 1, 1) \preceq (2, 2, 1) \preceq (3, 2) \preceq (4, 1) \preceq (5), \quad (4.34)$$

with respect to the dominance order. One should note that not every two elements of a partially ordered set are necessarily comparable through the partial order defined on the set. For example consider the set $\mathrm{Par}(6)$ together with dominance order $\preceq$. For partitions $\boldsymbol{\lambda} = (3, 1, 1, 1)$ and $\boldsymbol{\mu} = (2, 2, 2)$ in $\mathrm{Par}(6)$, none of the relations $\boldsymbol{\lambda} \preceq \boldsymbol{\mu}$ and $\boldsymbol{\mu} \preceq \boldsymbol{\lambda}$ is satisfied.

Despite this, if a finite set is partially ordered by a relation $\mathcal{R}$, it is always possible to arrange *all* elements of the set in a row such that the arrangement

is compatible with the partial order $\mathcal{R}$ in the sense that no element y is $\mathcal{R}$-related to an element x on the left hand side of y. To see this, let $\mathcal{R}$ be a partial order on a finite set A and let A_1 be the subset of A such that no elements of A_1 is $\mathcal{R}$-related to some element of A. Since A is assumed to be finite, A_1 is not empty. It is clear that elements of A_1 are not comparable to each other through $\mathcal{R}$. Now consider the complement of A_1 in A, that is, $A - A_1$. If $A - A_1$ is empty then any arrangement of the elements of A_1 is an arrangement of the elements of A compatible with $\mathcal{R}$. Otherwise, let A_2 be the subset of $A - A_1$ such that no elements of A_2 is $\mathcal{R}$-related to some element of $A - A_1$. Again A_2 is not empty and none of its elements are comparable with respect to $\mathcal{R}$. Now juxtaposition of an arrangement of elements of A_2 on the left hand side of an arrangement of elements of A_1 is still compatible with $\mathcal{R}$. For the next step, we consider $(A - A_1) - A_2$ and if this is not empty we consider the subset A_3 of $(A - A_1) - A_2$ and continue as before. Assume that after n steps the process ends. Then juxtaposition of any arrangement of elements of these subsets in the order $A_n, A_{n-1}, \ldots, A_1$ is an arrangement of elements of the original set A compatible with $\mathcal{R}$. By construction and the transitivity property of $\mathcal{R}$ it is clear that, if $i < j$, an element x_i in A_i is whether not comparable to an element x_j in A_j or x_j is $\mathcal{R}$-related to x_i. Of course this compatible arrangement is not in general unique.

If besides properties (ii) and (iii) on the previous page, a relation $\mathcal{R}$ on a set A is

(i′) *total*, that is, for any x and y in A, $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$,

it is called a *total* order. One can check that for $n \leqslant 5$, the dominance order on $\mathrm{Par}(n)$ is actually a total order. The elements of a totally ordered set can be uniquely arranged in a row to be compatible with the total order defined on the set in the sense that any element in the row is related to every each element on its right hand side.

4.2.4 Generating Function and the Number of Partitions

The number of partitions of a non-negative integer n is denoted by $p(n)$. There is no simple closed formula expressing $p(n)$ in terms of n, but one can write a *generating function* for the sequence $\big(p(k)\big)_{k \geqslant 0}$. The generating function $f(q)$ of a sequence $(a_k)_{k \geqslant 0}$ is defined to be the formal power series

$$f(q) = \sum_{k \geqslant 0} a_k q^k. \tag{4.35}$$

Formal here means that manipulations on these series such as summing and multiplying them together can be formally done without being concerned with convergence of the involved series. As an example, the generating function of

the constant sequence $1, 1, 1, \ldots$, is

$$f(q) = \sum_{k \geqslant 0} q^k = \frac{1}{1-q}, \tag{4.36}$$

regardless of the value of the variable q.

Now consider the function $F(q)$ defined by

$$F(q) = \prod_{i \geqslant 1} \sum_{m_i \geqslant 0} q^{im_i}. \tag{4.37}$$

Expanding the product, $F(q)$ can be written as

$$F(q) = \sum q^{m_1 + 2m_2 + 3m_3 + \cdots}, \tag{4.38}$$

where the sum is over all non-negative integer values of m_i's. Hence, for a given non-negative integer k, the coefficient of q^k in the sum above is exactly equal to the number of distinct sequences $(m_1, m_2, m_3, \ldots)$ consisting of non-negative integers such that $\sum_{i \geqslant 1} im_i = k$. By Equation (4.30), this is the number of partitions of k. Consequently, $F(q)$ is the generating function for the sequence $\big(p(k)\big)_{k \geqslant 0}$ and, therefore,

$$F(q) = \sum_{k \geqslant 0} p(k)q^k. \tag{4.39}$$

Using geometric series formula in Equation (4.37), one gets

$$\sum_{k \geqslant 0} p(k)q^k = \prod_{i \geqslant 1} \frac{1}{1-q^i}. \tag{4.40}$$

Analogously, the function

$$G(q) = \prod_{i=1}^{r} \sum_{m_i \geqslant 0} q^{im_i}, \tag{4.41}$$

is the generating function for the sequence $(p(k,r))_{k \geqslant 0}$, where $p(k,r)$ denotes the number of partitions of k with each part at most r or, equivalently, the number of partitions of k with at most r parts[§].

Of particular interest in this thesis is the number of partitions with at most n parts such that each part is at most d. As is explained shortly, this number is equal to $\binom{n+d}{n}$. In other words

$$\mathrm{Card}\left\{\, \boldsymbol{\lambda} \in \mathscr{P} \mid \lambda_1 \leqslant d \,, \, l(\boldsymbol{\lambda}) \leqslant n \,\right\} = \binom{n+d}{n}. \tag{4.42}$$

One way to see this it to establish a one-to-one correspondence between the set of partitions that fit into a rectangle of height n and width d and the set
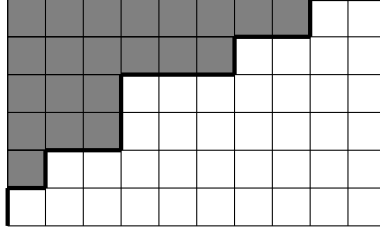
Figure 4.3: Gray squares represent $\boldsymbol{\lambda} = (8, 6, 3, 3, 1)$ inside a 6 by 10 rectangle and the polygon-line associated with it consists of six vertical and ten horizontal segments.

of polygon-lines consisting of n vertical (V) and d horizontal (H) segments, as is shown in Figure 4.3 for a special case of $n = 6$ and $d = 10$. Therefore, one can count the number of these polygon-lines instead of counting the number of mentioned partitions. It is clear that each such polygon-line consists of n vertical and d horizontal segments. Consequently, the number of these polygon-lines is equal to the number of permutations of a string of letters consisting of n letters of V and d letters of H, which is $\binom{n+d}{n}$ by a simple combinatorial argument.

Equation (4.42) can be viewed from a different angle. Let $p(k, d, n)$ be the number of partitions of k that fit into a rectangle of width d and height n. On page 33 of [And84] it is shown that the generating function for the sequence $\big(p(k, d, n)\big)_{k \geqslant 0}$ is the following function

$$\begin{aligned} G(q, d, n) &= \frac{(1 - q^{d+n})(1 - q^{d+n-1}) \cdots (1 - q^{n+1})}{(1 - q^d)(1 - q^{d-1}) \cdots (1 - q)} \\ &= \frac{(q)_{d+n}}{(q)_d (q)_n}, \end{aligned} \tag{4.43}$$

where for any non-negative integer m,

$$(q)_m := (1 - q^m)(1 - q^{m-1}) \cdots (1 - q). \tag{4.44}$$

Therefore,

$$\sum_{k \geqslant 0} p(k, d, n) q^k = \frac{(q)_{d+n}}{(q)_d (q)_n}. \tag{4.45}$$

Taking limit of both sides of Equation (4.45) as q goes to 1 and noting that $p(k, d, n) = 0$ for all integers k greater than dn, one gets

$$\sum_{k=0}^{dn} p(k, d, n) = \binom{n + d}{n}, \tag{4.46}$$

---

§See the paragraph followed by Equation (4.32).

and Equation (4.42) is recovered.

Finally, the *complement* of a partition $\boldsymbol{\lambda}$ with respect to a rectangle of height n and width d, which we denote by $\bar{\boldsymbol{\lambda}}$, is defined as

$$\bar{\boldsymbol{\lambda}} = (d - \lambda_n, d - \lambda_{n-1}, \ldots, d - \lambda_1). \tag{4.47}$$

For example, the complement of $\boldsymbol{\lambda} = (8, 6, 3, 3, 1)$ with respect to the rectangle shown in Figure 4.3 is $\bar{\boldsymbol{\lambda}} = (10, 9, 7, 7, 4, 2)$. It is clear that $l(\bar{\boldsymbol{\lambda}}) \leqslant n$ and $\bar{\lambda}_1 \leqslant d$.

4.3   Basic Homogeneous Symmetric Functions

This section introduces four of the most important homogeneous symmetric functions and their properties that are frequently used throughout this thesis. This includes monomial symmetric functions, elementary symmetric functions, complete homogeneous symmetric functions, and finally power-sum symmetric functions. All of them are homogeneous as well, although only the third one carries the name explicitly.

For simplicity, in the coming subsections one more piece of notation is employed. Let $\boldsymbol{x} = (x_1, x_2, x_3, \ldots)$ be an arrangement of variables and let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, , \alpha_3 \ldots)$ be a sequence of non-negative integers with α_n being the last non-zero term. Then

$$\boldsymbol{x}^{\boldsymbol{\alpha}} := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}. \tag{4.48}$$

If $\boldsymbol{\alpha}$ contains no non-zero term, then $\boldsymbol{x}^{\boldsymbol{\alpha}}$ is defined to be one. For example, $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1 x_4^3 x_5^2$ where $\boldsymbol{\alpha} = (1, 0, 0, 3, 2, 0, 0, 0, \ldots)$.

We also assume that the argument of any function, which is not mentioned explicitly, is $\boldsymbol{x}$.

4.3.1   Monomial Symmetric Functions

Let $\boldsymbol{\lambda}$ be a partition. The *monomial symmetric* function corresponding to $\boldsymbol{\lambda}$ is denoted by $m_{\boldsymbol{\lambda}}$ and it is defined by

$$m_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}, \tag{4.49}$$

where the sum is over all distinct permutations $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ of the terms in $\boldsymbol{\lambda}$ including tail-zeros. For example $m_{\emptyset} = 1$ and

$$m_{(1)} = \sum_i x_i, \tag{4.50}$$

$$m_{(1,1)} = \sum_{i_1 < i_2} x_{i_1} x_{i_2}, \tag{4.51}$$

$$m_{(2,1)} = \sum_{i_1 < i_2} (x_{i_1}^2 x_{i_2} + x_{i_1} x_{i_2}^2). \tag{4.52}$$

It is clear that for a partition $\boldsymbol{\lambda}$ of D, the function $m_{\boldsymbol{\lambda}}$ is a symmetric function of total degree D and degree λ_1 and belongs to $\Lambda_{,D}$ and also to Λ^d for all $d \geqslant \lambda_1$. It is also clear that if $f(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$ is a function in $\Lambda_{,D}$, then

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{\lambda} \vdash D} c_{\boldsymbol{\lambda}} m_{\boldsymbol{\lambda}}. \tag{4.53}$$

Therefore the set

$$\big\{\, m_{\boldsymbol{\lambda}} \mid \lambda \vdash D \,\big\} \tag{4.54}$$

is a basis for $\Lambda_{,D}$ and, as a result,

$$\dim(\Lambda_{,D}) = p(D). \tag{4.55}$$

Here $p(D)$ is the number of partitions of D. From Equation (4.20) it becomes clear that the set

$$\big\{\, m_{\boldsymbol{\lambda}} \mid \lambda \in \mathscr{P} \,\big\} \tag{4.56}$$

is a basis for Λ.

Now consider the following so-called *reduction* map ρ_n of Λ to Λ_n:

$$\rho_n(s)(\boldsymbol{x}) = s(x_1, \ldots, x_n, 0, 0, \ldots). \tag{4.57}$$

The monomial symmetric *polynomial* in n variables x_1 till x_n corresponding to partition $\boldsymbol{\lambda}$ is denoted explicitly by $m_{\boldsymbol{\lambda}}(x_1, \ldots, x_n)$ and it is defined by[§]

$$m_{\boldsymbol{\lambda}}(x_1, \ldots, x_n) = \rho_n(m_{\boldsymbol{\lambda}})(\boldsymbol{x}). \tag{4.58}$$

For example,

$$m_{(2,1,1)}(x_1, x_2, x_3) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2,$$
$$m_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2,$$
$$m_{(2,1,1)}(x_1, x_2) = 0.$$

Note that, by definition, $m_{\boldsymbol{\lambda}}(x_1, \ldots, x_n) = 0$ if and only if $l(\boldsymbol{\lambda}) > n$. It is clear that

$$\big\{\, m_{\boldsymbol{\lambda}}(x_1, \ldots, x_n) \mid \boldsymbol{\lambda} \vdash D \,,\, l(\boldsymbol{\lambda}) \leqslant n \,\big\}, \tag{4.59}$$

is a basis for $\Lambda_{n,D}$ and the set

$$\big\{\, m_{\boldsymbol{\lambda}}(x_1, \ldots, x_n) \mid l(\boldsymbol{\lambda}) \leqslant n \,\big\}, \tag{4.60}$$

is a basis for Λ_n. It is also clear that the set

$$\big\{\, m_{\boldsymbol{\lambda}}(x_1, \ldots, x_n) \mid \lambda_1 \leqslant d \,,\, l(\boldsymbol{\lambda}) \leqslant n \,\big\}, \tag{4.61}$$

[§]The same notation $m_{\boldsymbol{\lambda}}$ is used to denote both the monomial symmetric function and the monomial symmetric polynomial. To resolve this ambiguity, we agree to mention the argument $(x_1, \ldots, x_n)$ explicitly whenever dealing with a polynomial in a finite number of variables.

forms a basis for Λ_n^d. Therefore, the dimension of Λ_n^d is the number of partitions of n that fit into a rectangle of height n and width d. Thus by Equation (4.42),

$$\dim\left(\Lambda_n^d\right) = \binom{n+d}{n}. \tag{4.62}$$

Note that any symmetric polynomial in Λ_n is the image of a polynomial in Λ under the action of ρ_n. To see this, consider a polynomial $s(x_1, \ldots, x_n)$ in Λ_n. Since the set (4.54) is a basis for Λ_n, for some rational numbers $a_{\boldsymbol{\lambda}}$ one can write

$$s(x_1, \ldots, x_n) = \sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}}\, m_{\boldsymbol{\lambda}}(x_1, \ldots, x_n), \tag{4.63}$$

where the sum is over all partitions $\boldsymbol{\lambda}$ such that $l(\boldsymbol{\lambda}) \leqslant n$. Then it is obvious that, for the same partitions $\boldsymbol{\lambda}$, the image of $\sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}}\, m_{\boldsymbol{\lambda}}$ as a symmetric function in Λ under ρ_n is $s(x_1, \ldots, x_n)$.

4.3.2   Elementary Symmetric Functions

Given a partition $\boldsymbol{\lambda}$, the *elementary symmetric* function corresponding to $\boldsymbol{\lambda}$ is denoted by $e_{\boldsymbol{\lambda}}$ and it is defined by

$$e_{\boldsymbol{\lambda}} = e_{\lambda_1} e_{\lambda_2} e_{\lambda_3} \cdots, \tag{4.64}$$

where,

$$e_0 := 1, \tag{4.65}$$

and for any positive integer r, the rth elementary symmetric function e_r is defined by

$$e_r = m_{(1^r)} = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}. \tag{4.66}$$

Evidently, the total degree of $e_{\boldsymbol{\lambda}}$ is $|\boldsymbol{\lambda}|$ and its degree is $l(\boldsymbol{\lambda})$.

As the next step, let us find out that how these new symmetric functions are related to the monomial symmetric functions. It is shown in [Sta99] that for any partition $\boldsymbol{\lambda}$ of D,

$$e_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu} \vdash D} M_{\boldsymbol{\lambda}\boldsymbol{\mu}}\, m_{\boldsymbol{\mu}}, \tag{4.67}$$

where $M_{\boldsymbol{\lambda}\boldsymbol{\mu}}$'s are non-negative integers[§]. It is also shown that if $\boldsymbol{\lambda}'$ and $\boldsymbol{\mu}$ are not comparable through the dominance order or if $\boldsymbol{\lambda}' \preceq \mu$, then $M_{\boldsymbol{\lambda}\boldsymbol{\mu}} = 0$ and, moreover, $M_{\boldsymbol{\lambda}\boldsymbol{\lambda}'} = 1$.

Now assume that $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2, \ldots, \boldsymbol{\lambda}_{p(D)}$ is a particular arrangement of elements of $\mathrm{Par}(D)$ that is compatible with the dominance order[¶]. Since, $\boldsymbol{\lambda} \preceq \boldsymbol{\mu}$ if and

---

[§]In fact, it is shown in [Sta99] that $M_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is the number of matrices $A = [a_{ij}]_{i,j \geqslant 1}$ with entries 0 and 1 such that for all i, $\sum_{j \geqslant 1} a_{ij} = \lambda_i$, and for all j, $\sum_{i \geqslant 1} a_{ij} = \mu_j$.

[¶]Such an arrangement exists as is discussed at the end of page 28 and at the beginning of page 29.

only if $\boldsymbol{\mu}' \preceq \boldsymbol{\lambda}'$ [Mac95], $\boldsymbol{\lambda}'_{p(D)}, \ldots, \boldsymbol{\lambda}'_2, \boldsymbol{\lambda}'_1$ is also compatible with the dominance order. Now if one defines a $p(D)$-dimensional square matrix $\mathcal{M} = [\mathcal{M}_{ij}]$ by

$$\mathcal{M}_{ij} = M_{\boldsymbol{\lambda}_i \boldsymbol{\lambda}'_j}, \tag{4.68}$$

then from the lines followed by Equation (4.67) it is seen that $\mathcal{M}$ is an upper-triangular matrix with only 1 on the main diagonal and, consequently, it is invertible. For instance, for the $D = 4$ case we have

$$\begin{aligned}
e_{(1,1,1,1)} &= m_{(4)} + 4m_{(3,1)} + 6m_{(2,2)} + 12m_{(2,1,1)} + 24m_{(1,1,1,1)}, \\
e_{(2,1,1)} &= \qquad\quad m_{(3,1)} + 2m_{(2,2)} + 5m_{(2,1,1)} + 12m_{(1,1,1,1)}, \\
e_{(2,2)} &= \qquad\qquad\qquad\quad m_{(2,2)} + 2m_{(2,1,1)} + 6m_{(1,1,1,1)}, \\
e_{(3,1)} &= \qquad\qquad\qquad\qquad\qquad\quad m_{(2,1,1)} + 4m_{(1,1,1,1)}, \\
e_{(4)} &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad m_{(1,1,1,1)}.
\end{aligned}$$

This means that the transition matrix from the basis set (4.54) to the set

$$\left\{ e_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \vdash D \right\}, \tag{4.69}$$

is an invertible matrix. This implies that the set (4.69) also forms a basis for $\Lambda_{,D}$. Therefore, the set

$$\left\{ e_{\boldsymbol{\lambda}} \mid \lambda \in \mathscr{P} \right\} \tag{4.70}$$

forms a basis for Λ and, as a result, the set

$$\left\{ e_1, e_2, e_3, \ldots \right\} \tag{4.71}$$

is an algebraically independent generating set for Λ[§]. Thus any polynomial in Λ can be uniquely expressed as a polynomial in terms of $e_1, e_2, e_3, \ldots$. The last two statements constitute the content of what is known as the Fundamental Theorem of Symmetric Functions (FTSF), and the lines above sketched an outline of the proof. This theorem also implies that

$$\left\{ e_{\boldsymbol{\lambda}} \mid l(\boldsymbol{\lambda}) \leqslant d \right\} \tag{4.72}$$

forms a basis for Λ^d.

Before investigating the elementary symmetric polynomials in finite number of variables, let us explore the generating function for the sequence $(e_k)_{k \geqslant 0}$ that one would need in Section 4.3.4. Consider the function[¶]

$$E_{\boldsymbol{x}}(q) = \prod_{i \geqslant 1}(1 + x_i q), \tag{4.73}$$

[§]The identity polynomial $i(\boldsymbol{x}) = 1$ is generated by this set through the identity polynomial itself, namely, $i(e_1, e_2, \ldots) = i$.

[¶]as a polynomial in just one single variable q and infinite number of real parameters x_1, x_2, x_3, $\ldots$. Actually, $E_{\boldsymbol{x}}(q)$ is a member of $\Lambda_1[q]$.

and expand the product on the right hand side of this equation. For a given non-negative integer k, the coefficient of q^k in this expansion is e_k and, therefore,

$$\sum_{k \geqslant 0} e_k q^k = \prod_{i \geqslant 1}(1 + x_i q). \tag{4.74}$$

Thus, $E_x(q)$ in Equation (4.73) is the generating function for the sequence $(e_k)_{k \geqslant 0}$.

Let us now consider the case of the finite number of variables. For a non-negative integer r, the rth elementary symmetric *polynomial* in variables x_1 till x_n is denoted explicitly by $e_r(x_1, \ldots, x_n)$ and it is defined by

$$e_r(x_1, \ldots, x_n) = \rho_n(e_r)(\boldsymbol{x}), \tag{4.75}$$

where ρ_n is the reduction map defined by Equation (4.57). For instance,

$$e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3, \tag{4.76}$$
$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 + x_3 x_1, \tag{4.77}$$
$$e_3(x_1, x_2, x_3) = x_1 x_2 x_3, \tag{4.78}$$
$$e_n(x_1, x_2, x_3) = 0, \tag{4.79}$$

for all integers $n \geqslant 4$. The definition of $e_{\boldsymbol{\lambda}}(x_1, \ldots, x_n)$ is the same as its infinite-case counterpart. For example

$$\begin{aligned} e_{(2,1,1)}(x_1, x_2) &= e_2(x_1, x_2)e_1(x_1, x_2)e_1(x_1, x_2) \\ &= x_1 x_2 (x_1 + x_2)^2. \end{aligned} \tag{4.80}$$

By definition it is readily seen that $e_{\boldsymbol{\lambda}}(x_1, \ldots, x_n) = 0$ if and only if $\lambda_1 > n$.

Based on triangularity property described in the FTSF, the set

$$\left\{ e_{\boldsymbol{\lambda}}(x_1, \ldots, x_n) \mid \lambda_1 \leqslant n , \ \boldsymbol{\lambda} \vdash D \right\}, \tag{4.81}$$

is a basis for $\Lambda_{n,D}$ and the set

$$\left\{ e_{\boldsymbol{\lambda}}(x_1, \ldots, x_n) \mid \lambda_1 \leqslant n \right\}, \tag{4.82}$$

is a basis for Λ_n. Therefore,

$$\left\{ e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n) \right\} \tag{4.83}$$

is an algebraically independent generating set for Λ_n. Finally the set

$$\left\{ e_{\boldsymbol{\lambda}}(x_1, \ldots, x_n) \mid \lambda_1 \leqslant n , \ l(\boldsymbol{\lambda}) \leqslant d \right\} \tag{4.84}$$

is a basis for Λ_n^d.

4.3.3 Complete Homogeneous Symmetric Functions

Given a partition $\boldsymbol{\lambda}$, the *complete homogeneous symmetric* function corresponding to $\boldsymbol{\lambda}$ is denoted by $h_{\boldsymbol{\lambda}}$ and it is defined by

$$h_{\boldsymbol{\lambda}} = h_{\lambda_1} h_{\lambda_2} h_{\lambda_3} \cdots , \tag{4.85}$$

where

$$h_0 := 1, \tag{4.86}$$

and for any positive integer r, the rth complete homogeneous symmetric function h_r is defined by

$$h_r = \sum_{\boldsymbol{\lambda} \vdash r} m_{\boldsymbol{\lambda}}. \tag{4.87}$$

It is clear that the total degree and the degree of $h_{\boldsymbol{\lambda}}$ are both equal to $|\boldsymbol{\lambda}|$.

There is a relation similar to Equation (4.67) that enables us to write $h_{\boldsymbol{\lambda}}$'s as a linear combination of monomial symmetric functions. It is shown in [Sta99] that for any partition $\boldsymbol{\lambda}$ of D,

$$h_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\lambda} \vdash D} N_{\boldsymbol{\lambda}\boldsymbol{\mu}} \, m_{\boldsymbol{\mu}}, \tag{4.88}$$

where $N_{\boldsymbol{\lambda}\boldsymbol{\mu}}$'s are non-negative integers[§]. For example, for $D = 4$ case

$$
\begin{aligned}
h_{(1,1,1,1)} &= 24 m_{(1,1,1,1)} + 12 m_{(2,1,1)} + 6 m_{(2,2)} + 4 m_{(3,1)} + m_{(4)}, \\
h_{(2,1,1)} &= 12 m_{(1,1,1,1)} + 7 m_{(2,1,1)} + 4 m_{(2,2)} + 3 m_{(3,1)} + m_{(4)}, \\
h_{(2,2)} &= 6 m_{(1,1,1,1)} + 4 m_{(2,1,1)} + 3 m_{(2,2)} + 2 m_{(3,1)} + m_{(4)}, \\
h_{(3,1)} &= 4 m_{(1,1,1,1)} + 3 m_{(2,1,1)} + 2 m_{(2,2)} + 2 m_{(3,1)} + m_{(4)}, \\
h_{(4)} &= m_{(1,1,1,1)} + m_{(2,1,1)} + m_{(2,2)} + m_{(3,3)} + m_{(4)}.
\end{aligned}
$$

Thus, as is clear from example above, the transition matrix from the basis set (4.54) to the set

$$\left\{ \, h_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \vdash D \, \right\} \tag{4.89}$$

is no longer triangular as in the case of elementary symmetric polynomials. Hence to prove that the set above forms a basis for $\Lambda_{,D}$ one needs to follow a different strategy, which is the subject of Section 4.4. When this is done, one can see that the set

$$\left\{ \, h_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P} \, \right\} \tag{4.90}$$

constitutes a basis for Λ and therefore,

$$\left\{ \, h_1, h_2, h_3, \ldots \, \right\} \tag{4.91}$$

is an algebraically independent generating set for Λ.

[§]Actually, $N_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ is the number of matrices $A = [a_{ij}]_{i,j \geqslant 1}$ with non-negative-integer entries such that for all i, $\sum_{j \geqslant 1} a_{ij} = \lambda_i$, and for all j, $\sum_{i \geqslant 1} a_{ij} = \mu_j$.

To find out the generating function for the sequence $(h_k)_{k \geqslant 0}$, consider the following function

$$H_{\boldsymbol{x}}(q) = \prod_{i \geqslant 1} \frac{1}{1 - x_i q}, \tag{4.92}$$

as a function in $\Lambda_1[q]$ for infinite number of parameters $x_1, x_2, x_3, \ldots$. Using geometric series formula, Equation (4.92) can be written as

$$\begin{aligned}
H_{\boldsymbol{x}}(q) &= \prod_{i \geqslant 1} \left(\sum_{r_i \geqslant 0} x_i^{r_i} q^{r_i} \right) \\
&= \sum_{r_1 \geqslant 0} \sum_{r_2 \geqslant 0} \cdots \left(x_1^{r_1} x_2^{r_2} \cdots q^{r_1 + r_2 + \cdots} \right),
\end{aligned} \tag{4.93}$$

and, therefore, for a given non-negative integer k, the coefficient of q^k is

$$\sum \left(x_1^{r_1} x_2^{r_2} \cdots \right). \tag{4.94}$$

Here the sum is over all non-negative integers $r_1, r_2, \ldots$ such that

$$\sum_{i \geqslant 1} r_i = k. \tag{4.95}$$

Thus, expression (4.94) is equal to

$$\sum_{\boldsymbol{\mu} \vdash k} m_{\boldsymbol{\mu}} = h_k, \tag{4.96}$$

and $H_x(q)$ is the desired generating function. In other words

$$\sum_{k \geqslant 0} h_k q^k = \prod_{i \geqslant 1} \frac{1}{1 - x_i q}. \tag{4.97}$$

The complete homogeneous symmetric *polynomial* $h_{\boldsymbol{\lambda}}(x_1, \ldots, x_n)$ corresponding to a given partition $\boldsymbol{\lambda}$ is defined as for the previous cases through the reduction map ρ_n. For a non-negative integer r, the rth complete homogeneous symmetric polynomial in n variables x_1 till x_n is denoted explicitly by $h_r(x_1, \ldots, x_n)$ and it is defined by

$$h_r(x_1, \ldots, x_n) = \rho_n(h_r)(\boldsymbol{x}). \tag{4.98}$$

For example,

$$\begin{aligned}
h_3(x_1, x_2) &= m_{(3)}(x_1, x_2) + m_{(2,1)}(x_1, x_2) + m_{(1,1,1)}(x_1, x_2) \\
&= x_1^3 + x_2^3 + x_1^2 x_2 + x_1 x_2^2,
\end{aligned} \tag{4.99}$$

and

$$h_2(x_1, x_2) = m_{(2)}(x_1, x_2) + m_{(1,1)}(x_1, x_2)$$
$$= x_1^2 + x_2^2 + x_1 x_2, \tag{4.100}$$

therefore,

$$h_{(3,2)}(x_1, x_2) = (x_1^3 + x_2^3 + x_1^2 x_2 + x_1 x_2^2)(x_1^2 + x_2^2 + x_1 x_2). \tag{4.101}$$

In Section 4.4 it is shown that the set

$$\left\{ h_{\boldsymbol{\lambda}}(x_1, \ldots, x_n) \mid \lambda_1 \leqslant n , \ \boldsymbol{\lambda} \vdash D \right\} \tag{4.102}$$

forms a basis for $\Lambda_{n,D}$ and, consequently, the set

$$\left\{ h_{\boldsymbol{\lambda}}(x_1, \ldots, x_n) \mid \lambda_1 \leqslant n \right\} \tag{4.103}$$

constitutes a basis for Λ_n. Hence

$$\left\{ h_1(x_1, \ldots, x_n), \ldots, h_n(x_1, \ldots, x_n) \right\} \tag{4.104}$$

is an algebraically independent generating set for Λ_n.

4.3.4 Power-sum Symmetric Functions

Let $\boldsymbol{\lambda}$ be a partition. The *power-sum symmetric* function corresponding to $\boldsymbol{\lambda}$ is denoted by $p_{\boldsymbol{\lambda}}$ and it is defined by

$$p_{\boldsymbol{\lambda}} = p_{\lambda_1} p_{\lambda_2} \cdots , \tag{4.105}$$

where,

$$p_0 := 1, \tag{4.106}$$

and for any positive integer r, the rth power-sum symmetric function p_r is defined by

$$p_r = m_{(r)} = \sum_{i \geqslant 1} x_i^r. \tag{4.107}$$

As in the case of complete homogeneous symmetric functions, the total degree and the degree of $p_{\boldsymbol{\lambda}}$ are both equal to $|\boldsymbol{\lambda}|$.

Now we want to determine a basis for $\Lambda_{,D}$ and Λ composed of power-sum symmetric functions. To this end, we first determine the generating function $P_x(q)$ for the sequence[§] $(p_k)_{k \geqslant 1}$. By the definition of the generating function of a sequence

$$P_{\boldsymbol{x}}(q) = \sum_{k \geqslant 1} p_k q^{k-1}. \tag{4.108}$$

[§]Note that the starting index is 1 rather than 0.

Thus using the defining Equation (4.107) and the geometric series formula

$$P_{\boldsymbol{x}}(q) = \sum_{k \geqslant 1} \left(\sum_{i \geqslant 1} x_i^k \right) q^{k-1}$$
$$= \sum_{i \geqslant 1} \frac{x_i}{1 - x_i q}$$
$$= \sum_{i \geqslant 1} \frac{\mathrm{d}}{\mathrm{d}q} \ln \left(\frac{1}{1 - x_i q} \right)$$
$$= \frac{\mathrm{d}}{\mathrm{d}q} \ln \left(\prod_{i \geqslant 1} \frac{1}{1 - x_i q} \right), \tag{4.109}$$

or, using Equation (4.92),

$$P_{\boldsymbol{x}}(q) = \frac{H_{\boldsymbol{x}}'(q)}{H_{\boldsymbol{x}}(q)}. \tag{4.110}$$

Equations (4.73) and (4.92) give

$$H_{\boldsymbol{x}}(q) = \frac{1}{E_{\boldsymbol{x}}(-q)} \tag{4.111}$$

and, therefore,

$$H_{\boldsymbol{x}}'(q) = E_{\boldsymbol{x}}'(-q) H_{\boldsymbol{x}}^2(q). \tag{4.112}$$

Using Equations (4.111) and (4.112) in Equation (4.110) yields

$$P_{\boldsymbol{x}}(-q) E_{\boldsymbol{x}}(q) = E_{\boldsymbol{x}}'(q), \tag{4.113}$$

and from there, by substituting power-series expansions of $P_x(-q)$ and $E_x(q)$, we get

$$\left(\sum_{s \geqslant 0} (-1)^s p_{s+1} q^s \right) \left(\sum_{t \geqslant 0} e_t q^t \right) = \sum_{m \geqslant 0} (m+1) e_{m+1} q^m. \tag{4.114}$$

Comparing the coefficients of q^{k-1} on both sides of Equation (4.114), gives

$$k e_k = \sum_{r=1}^{k} (-1)^{r-1} p_r e_{k-r}, \qquad (k \geqslant 1). \tag{4.115}$$

These are called *Newton's identities* that establish a relation between e's and p's. Cramer's rule can be exploited to solve e's in terms of p's. Considering e_1 till e_n to be unknowns in the first n Newton's identities, Cramer's rule yields

$$e_n = \frac{1}{n!} \begin{vmatrix} p_1 & 1 & 0 & \dots & 0 \\ p_2 & p_1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ p_{n-1} & p_{n-2} & \dots & p_1 & n-1 \\ p_n & p_{n-1} & \dots & p_2 & p_1 \end{vmatrix}. \tag{4.116}$$

Reversely, by considering p_1 till p_n to be unknowns, these equations give

$$p_n = \begin{vmatrix} e_1 & 1 & 0 & \dots & 0 \\ 2e_2 & e_1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ (n-1)e_{n-1} & e_{n-2} & \dots & e_1 & 1 \\ ne_n & e_{n-1} & \dots & e_2 & e_1 \end{vmatrix}. \tag{4.117}$$

Similarly, by plugging power-series expansions of $P_x(q)$ and $H_x(q)$ in Equation (4.110) and comparing the coefficients of q^k for a given positive integer k, yields

$$kh_k = \sum_{r=1}^{k} p_r h_{k-r}, \qquad (k \geqslant 1), \tag{4.118}$$

that connects h's and p's. Using Cramer's rule one can also write Equations (4.118) in the form of determinants.

As discussed in Section 4.3.2, $\{e_1, e_2, e_3, \dots\}$ is a generating set for Λ. From Equations (4.115), by iterative calculations, each e_k can be written as a polynomial in terms of p_1 till p_k with rational coefficients. Hence, the set $\{p_1, p_2, p_3, \dots\}$ also generates Λ. On the other hand, Newton's identities also enable us to express a given p_k in terms of e_1 till e_k as a polynomial with rational (integer, to be more precise) coefficients and, therefore, since $\{e_1, e_2, e_3, \dots\}$ is algebraically independent, so is $\{p_1, p_2, p_3, \dots\}$. This is equivalent to saying that the set

$$\{p_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P}\} \tag{4.119}$$

is a basis for Λ. So according to Equation (4.20), the set

$$\{p_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \vdash D\} \tag{4.120}$$

is a basis for $\Lambda_{,D}$.

That the sets (4.119) and (4.120) form bases for Λ and $\Lambda_{,D}$, respectively, can be viewed from a different angle. In [Sta99], it is shown that for a partition $\boldsymbol{\lambda}$ of D,

$$p_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\mu} \vdash D} R_{\boldsymbol{\lambda}\boldsymbol{\mu}} \, m_{\boldsymbol{\mu}}, \tag{4.121}$$

for some non-negative integers $R_{\boldsymbol{\lambda}\boldsymbol{\mu}}$ such that $R_{\boldsymbol{\lambda}\boldsymbol{\mu}} = 0$ unless $\boldsymbol{\lambda} \preceq \boldsymbol{\mu}$ and

$$R_{\boldsymbol{\lambda}\boldsymbol{\lambda}} = \prod_{i \geqslant 1} m_i(\boldsymbol{\lambda})!. \tag{4.122}$$

Therefore the transition matrix from the basis set (4.54) to the set (4.120) is triangular with non-zero diagonal entries that implies the sets above form bases for Λ and $\Lambda_{,D}$, respectively.

As the final point of this section, we consider the case of finite number of variables and introduce the power-sum symmetric polynomial $p_{\boldsymbol{\lambda}}(x_1, \ldots, x_n)$ that corresponds to the partition $\boldsymbol{\lambda}$. This is done as in the case of symmetric polynomials in the previous sections, namely, through the reduction map ρ_n. The rth power-sum symmetric polynomial $p_r(x_1, \ldots, x_n)$ is defined by

$$p_r(x_1, \ldots, x_n) = \rho_n(p_r)(\boldsymbol{x}). \tag{4.123}$$

The only exception is that this time

$$p_0(x_1, \ldots, x_n) := n. \tag{4.124}$$

Based on triangularity of the R matrix introduced by Equation (4.121), it is shown in [Sta99] that the set

$$\left\{ p_{\boldsymbol{\lambda}}(x_1, \ldots, x_n) \mid \lambda_1 \leqslant n \right\} \tag{4.125}$$

is a basis for Λ_n and the set

$$\left\{ p_1(x_1, \ldots, x_n), \ldots, p_n(x_1, \ldots, x_n) \right\} \tag{4.126}$$

is algebraically independent. As a result, any polynomial in Λ_n can be uniquely expressed as a polynomial in the elements of this set.

4.4   An Involution on Λ

The goal of this section is to show that the sets introduced in (4.89) and (4.90) are in fact bases for $\Lambda_{,D}$ and Λ, respectively, as claimed there. To this end, we first define an involution on Λ. An *involution* on a set X is a map f of X onto itself such that $f \circ f = \mathrm{id}_X$, where id_X is the identity map on X. In other words, f is an involution if it is its own inverse. Therefore, any involution is a bijective map. For example, the function f defined on the set of real numbers by $f(x) = 1 - x$ is an involution.

We use the following strategy to do this. First we define an algebra endomorphism ω on Λ by defining how it acts on the elements of the algebraically independent generating set $\left\{ e_1, e_2, e_3, \ldots \right\}$ of Λ[§]. The definition of ω is engineered such that the image of the set (4.70) under the action of ω is the set (4.90). Then we show that ω is in fact an involution on Λ and, consequently, the image of any basis of Λ under ω is another basis of Λ.

The ω mentioned above is defined by

$$\begin{cases} \omega : \Lambda \longrightarrow \Lambda, \\ \omega(e_k) = h_k \end{cases}, \tag{4.127}$$

[§]Refer to the paragraph just above the Section 4.2.

for all positive integers k. Since Λ is an algebra without any zero divisor, $\omega(e_0) = h_0$ and therefore

$$\omega(e_{\boldsymbol{\lambda}}) = h_{\boldsymbol{\lambda}}, \tag{4.128}$$

for all partitions $\boldsymbol{\lambda}$ including the empty partition. As mentioned earlier, we want to show that ω is an involution on Λ. First, we determine how ω affects on h_k's. Writing Equation (4.111) as

$$E_{\boldsymbol{x}}(-q)H_{\boldsymbol{x}}(q) = 1, \tag{4.129}$$

and plugging the corresponding power series for $E_x(-q)$ and $H_x(q)$ into the equation above, yields

$$\sum_{k \geqslant 0} \left(\sum_{r=0}^{k} (-1)^r e_r h_{k-r} \right) q^k = 1, \tag{4.130}$$

and, consequently,

$$\sum_{r=0}^{k} (-1)^r e_r h_{k-r} = 0, \qquad (k \geqslant 1). \tag{4.131}$$

Applying ω on both sides of Equations (4.131), one gets

$$\sum_{r=0}^{k} (-1)^r h_r \omega(h_{k-r}) = 0, \qquad (k \geqslant 1), \tag{4.132}$$

and reversing the order of the terms in this sum gives rise to

$$\sum_{r=0}^{k} (-1)^r \omega(h_r) h_{k-r} = 0, \qquad (k \geqslant 1). \tag{4.133}$$

By writing Equations (4.131) and (4.133) explicitly for all positive integers k and comparing the resultant equations correspondingly and noting that

$$\omega(h_0) = \omega(e_0) = h_0 = 1, \tag{4.134}$$

one obtains

$$\omega(h_k) = e_k, \qquad (k \geqslant 0) \tag{4.135}$$

or, as a direct result,

$$\omega(h_{\boldsymbol{\lambda}}) = e_{\boldsymbol{\lambda}}. \tag{4.136}$$

Using of Equations (4.128) and (4.136), we get

$$\omega^2(e_{\boldsymbol{\lambda}}) = e_{\boldsymbol{\lambda}}. \tag{4.137}$$

This means that ω^2 acts as the identity map on the basis elements $e_{\boldsymbol{\lambda}}$ of Λ, that is,

$$\omega^2 = \mathrm{id}_\Lambda, \tag{4.138}$$

and ω is an involution on Λ. As becomes clear in the next section, ω defined in this way turns out to be a valuable tool to help us to prove Theorem 4.4.

Similarly one can define an algebra endomorphism

$$\begin{cases} \omega_n : \Lambda_n \longrightarrow \Lambda_n \\ \omega_n(e_k) = h_k \end{cases}, \tag{4.139}$$

for all positive integers k not greater than n[§]. Thus, $\omega_n(e_{\boldsymbol{\lambda}}) = h_{\boldsymbol{\lambda}}$ for all partitions $\boldsymbol{\lambda}$ with $\lambda_1 \leqslant n$. Analogously, one can show that $\omega_n^2(e_{\boldsymbol{\lambda}}) = e_{\boldsymbol{\lambda}}$ for all partitions $\boldsymbol{\lambda}$ with $\lambda_1 \leqslant n$. Since the set (4.82) is a basis for Λ_n, this yields

$$\omega_n^2 = \mathrm{id}_{\Lambda_n}, \tag{4.140}$$

and the set (4.103) is a basis for Λ_n.

4.5   The Main Theorem

This last section contains the proof for Theorem 4.4, which is the main point of this long chapter and is of particular interest in this thesis. To this end, some machinery must be developed first.

As the first step, we try to determine the effect of the involution ω, defined in the preceding section, on $p_{\boldsymbol{\lambda}}$ for a given partition $\boldsymbol{\lambda}$. Multiplying both sides of Equation (4.110) by $H_{\boldsymbol{x}}(q)$ and acting on both sides of the resulting equation by ω, yields

$$\omega\big(P_{\boldsymbol{x}}(q)\big)\omega\big(H_{\boldsymbol{x}}(q)\big) = \omega\big(H'_{\boldsymbol{x}}(q)\big). \tag{4.141}$$

One can employ Equation (4.135) and write

$$\begin{aligned} \omega\big(H_{\boldsymbol{x}}(q)\big) &= \sum_{k \geqslant 0} \omega(h_k)q^k \\ &= \sum_{k \geqslant 0} e_k q^k \\ &= E_{\boldsymbol{x}}(q). \end{aligned} \tag{4.142}$$

Similarly,

$$\omega\big(H'_{\boldsymbol{x}}(q)\big) = E'_{\boldsymbol{x}}(q). \tag{4.143}$$

Plugging the results of the last two equations in Equation (4.141) and comparing the resultant equation with Equation (4.113), yields

$$\omega\big(P_{\boldsymbol{x}}(q)\big) = P_{\boldsymbol{x}}(-q), \tag{4.144}$$

[§]Here e_k and h_k, although written without argument $(x_1, \ldots, x_n)$, refer to elements of Λ_n.

or, equivalently,

$$\sum_{k \geqslant 1} \omega(p_k) q^{k-1} = \sum_{k \geqslant 1} (-1)^{k-1} p_k q^{k-1}, \tag{4.145}$$

and, therefore, for all positive integers k,

$$\omega(p_k) = (-1)^{k-1} p_k. \tag{4.146}$$

For a given non-empty partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$, Equation (4.146) gives

$$\begin{aligned} \omega(p_{\boldsymbol{\lambda}}) &= \omega(p_{\lambda_1}) \cdots \omega(p_{\lambda_r}) \\ &= (-1)^{\lambda_1 + \cdots + \lambda_r - r} p_{\lambda_1} \cdots p_{\lambda_r}, \end{aligned} \tag{4.147}$$

which can be written as

$$\omega(p_{\boldsymbol{\lambda}}) = \varepsilon_{\boldsymbol{\lambda}} p_{\boldsymbol{\lambda}}, \tag{4.148}$$

where

$$\varepsilon_{\boldsymbol{\lambda}} = (-1)^{|\boldsymbol{\lambda}| - l(\boldsymbol{\lambda})}. \tag{4.149}$$

Equation (4.148) can be separately checked to be true for the empty partition.

As the second step, we derive an alternative expression for the generating functions $H_{\boldsymbol{x}}(q)$ and $E_{\boldsymbol{x}}(q)$. Solving Equation (4.110) as a differential equation for $H_{\boldsymbol{x}}(q)$ and noting that $H_{\boldsymbol{x}}(0) = 1$, we get

$$\begin{aligned} H_{\boldsymbol{x}}(q) &= \exp\left(\int_0^q P_{\boldsymbol{x}}(s) \, \mathrm{d}s \right) \\ &= \exp\left[\int_0^q \left(\sum_{k \geqslant 1} p_k s^{k-1} \right) \mathrm{d}s \right] \\ &= \exp\left(\sum_{k \geqslant 1} \frac{p_k q^k}{k} \right) \\ &= \prod_{k \geqslant 1} \exp\left(\frac{p_k q^k}{k} \right) \\ &= \prod_{k \geqslant 1} \sum_{r \geqslant 0} \frac{(p_k q^k)^r}{k^r r!}. \end{aligned} \tag{4.150}$$

Expanding the product above gives

$$\begin{aligned} H_{\boldsymbol{x}}(q) &= \sum_{r_1 \geqslant 0} \frac{(p_1 q^1)^{r_1}}{1^{r_1} r_1!} \sum_{r_2 \geqslant 0} \frac{(p_2 q^2)^{r_2}}{2^{r_2} r_2!} \sum_{r_3 \geqslant 0} \frac{(p_3 q^3)^{r_3}}{3^{r_3} r_3!} \cdots \\ &= \sum \frac{p_1^{r_1} p_2^{r_2} p_3^{r_3} \cdots}{1^{r_1} r_1! \, 2^{r_2} r_2! \, 3^{r_3} r_3! \cdots} q^{r_1 + 2r_2 + 3r_3 + \cdots}, \end{aligned} \tag{4.151}$$

where the last sum is over all possible distinct combinations of non-negative integers $r_1, r_2, r_3, \ldots$. Now let $\boldsymbol{\lambda}$ be a partition whose ith multiplicity $m_i(\boldsymbol{\lambda})$ equals r_i. Then

$$q^{r_1 + 2r_2 + 3r_3 + \cdots} = q^{|\boldsymbol{\lambda}|}, \tag{4.152}$$

and

$$p_1^{r_1} p_2^{r_2} p_3^{r_3} \cdots = p_{\boldsymbol{\lambda}}. \tag{4.153}$$

Moreover, when r_i's run through the set of all non-negative integers, $\boldsymbol{\lambda}$ runs through all partitions. Therefore, Equation (4.151) can be written in a more compact form as

$$H_{\boldsymbol{x}}(q) = \sum_{\boldsymbol{\lambda} \in \mathscr{P}} z_{\boldsymbol{\lambda}}^{-1} p_{\boldsymbol{\lambda}} q^{|\boldsymbol{\lambda}|}, \tag{4.154}$$

where

$$z_{\boldsymbol{\lambda}} = \prod_{i \geqslant 1} i^{m_i(\boldsymbol{\lambda})} \, m_i(\boldsymbol{\lambda})!. \tag{4.155}$$

For a given non-negative integer k, the coefficient of q^k in $H_{\boldsymbol{x}}(q)$ is h_k. Thus, from Equation (4.154) we get

$$h_k = \sum_{\boldsymbol{\lambda} \vdash k} z_{\boldsymbol{\lambda}}^{-1} p_{\boldsymbol{\lambda}}, \qquad (k \geqslant 0). \tag{4.156}$$

Applying ω on both sides of Equation (4.154) and exploiting Equations (4.142) and (4.148) results in

$$E_{\boldsymbol{x}}(q) = \sum_{\boldsymbol{\lambda} \in \mathscr{P}} \varepsilon_{\boldsymbol{\lambda}} z_{\boldsymbol{\lambda}}^{-1} p_{\boldsymbol{\lambda}} q^{|\boldsymbol{\lambda}|}, \tag{4.157}$$

and, therefore,

$$e_k = \sum_{\boldsymbol{\lambda} \vdash k} \varepsilon_{\boldsymbol{\lambda}} z_{\boldsymbol{\lambda}}^{-1} p_{\boldsymbol{\lambda}}, \qquad (k \geqslant 0). \tag{4.158}$$

Now we prove two lemmas that are not only interesting results on their own rights but also pave the way to prove Theorem 4.4. In what follows, $\boldsymbol{x} = (x_1, x_2, \ldots)$ and $\boldsymbol{y} = (y_1, y_2, \ldots)$.

Lemma 4.1. *The following two identities*

$$\prod_{i \geqslant 1} \prod_{j \geqslant 1} \frac{1}{1 - x_i y_j} = \sum_{\boldsymbol{\lambda} \in \mathscr{P}} z_{\boldsymbol{\lambda}}^{-1} p_{\boldsymbol{\lambda}}(\boldsymbol{x}) p_{\boldsymbol{\lambda}}(\boldsymbol{y}), \tag{4.159}$$

and

$$\prod_{i \geqslant 1} \prod_{j \geqslant 1} \frac{1}{1 - x_i y_j} = \sum_{\boldsymbol{\lambda} \in \mathscr{P}} h_{\boldsymbol{\lambda}}(\boldsymbol{x}) m_{\boldsymbol{\lambda}}(\boldsymbol{y}) = \sum_{\boldsymbol{\lambda} \in \mathscr{P}} m_{\boldsymbol{\lambda}}(\boldsymbol{x}) h_{\boldsymbol{\lambda}}(\boldsymbol{y}), \tag{4.160}$$

hold.

Proof. For the kth power-sum symmetric function p_k, using simple algebra one can show

$$p_k(\boldsymbol{w}) = p_k(\boldsymbol{x}) p_k(\boldsymbol{y}), \tag{4.161}$$

where $\boldsymbol{w} = (w_1, w_2, w_3, \ldots)$ is defined by

$$\boldsymbol{w} = (x_1 y_1, x_1 y_2, x_1 y_3, \ldots, x_2 y_1, x_2 y_2, x_2 y_3 \ldots, x_3 y_1, x_3 y_2, x_3 y_3, \ldots). \quad (4.162)$$

Consequently, for any partition $\boldsymbol{\lambda}$,

$$p_{\boldsymbol{\lambda}}(\boldsymbol{w}) = p_{\boldsymbol{\lambda}}(\boldsymbol{x}) p_{\boldsymbol{\lambda}}(\boldsymbol{y}). \quad (4.163)$$

Assuming $\boldsymbol{w}$ as given in Equation (4.162), the function $H_{\boldsymbol{w}}(q)$ defined by

$$H_w(q) = \prod_{s \geqslant 1} \frac{1}{1 - w_s q}, \quad (4.164)$$

is the generating function for the sequence $\big(h_k(\boldsymbol{w})\big)_{k \geqslant 0}$ and, moreover,

$$
\begin{aligned}
H_{\boldsymbol{w}}(1) &= \prod_{s \geqslant 1} \frac{1}{1 - w_s} \\
&= \prod_{i \geqslant 1} \prod_{j \geqslant 1} \frac{1}{1 - x_i y_j}.
\end{aligned} \quad (4.165)
$$

On the other hand, by Equations (4.154) and (4.163)

$$
\begin{aligned}
H_{\boldsymbol{w}}(q) &= \sum_{\boldsymbol{\lambda} \in \mathscr{P}} z_{\boldsymbol{\lambda}}^{-1} p_{\boldsymbol{\lambda}}(\boldsymbol{w}) q^{|\boldsymbol{\lambda}|} \\
&= \sum_{\boldsymbol{\lambda} \in \mathscr{P}} z_{\boldsymbol{\lambda}}^{-1} p_{\boldsymbol{\lambda}}(\boldsymbol{x}) p_{\boldsymbol{\lambda}}(\boldsymbol{y}) q^{|\boldsymbol{\lambda}|}.
\end{aligned} \quad (4.166)
$$

Thus

$$H_{\boldsymbol{w}}(1) = \sum_{\boldsymbol{\lambda} \in \mathscr{P}} z_{\boldsymbol{\lambda}}^{-1} p_{\boldsymbol{\lambda}}(\boldsymbol{x}) p_{\boldsymbol{\lambda}}(\boldsymbol{y}). \quad (4.167)$$

The proof for the first identity is done if one compares Equation (4.165) with Equation (4.167).

To prove the second identity, one notes that the left hand side of this identity is

$$\prod_{j=1}^{m} H_x(y_j), \quad (4.168)$$

where $H_{\boldsymbol{x}}(y_j)$ is the generating function for the sequence $\big(h_k(\boldsymbol{x})\big)_{k \geqslant 0}$ with y_j as its variable. Thus, the left hand side of the second identity can be written as

$$
\begin{aligned}
\prod_{i \geqslant 1} \prod_{j \geqslant 1} \frac{1}{1 - x_i y_j} &= \prod_{j \geqslant 1} \left(\sum_{k \geqslant 0} h_k(\boldsymbol{x}) y_j^k \right) \\
&= \left(\sum_{k_1 \geqslant 0} h_{k_1}(\boldsymbol{x}) y_1^{k_1} \right) \left(\sum_{k_2 \geqslant 0} h_{k_2}(\boldsymbol{x}) y_2^{k_2} \right) \cdots \\
&= \sum h_{k_1}(\boldsymbol{x}) h_{k_2}(\boldsymbol{x}) \cdots y_1^{k_1} y_2^{k_2} \cdots ,
\end{aligned} \quad (4.169)
$$

where the last sum is over all possible arrangements of non-negative integers $k_1, k_2, \ldots$ and, by definitions of h_λ and m_λ, it is equal to $\sum_{\lambda \in \mathscr{P}} h_\lambda(\boldsymbol{x}) m_\lambda(\boldsymbol{y})$. Interchanging x's and y's, one can prove the other part of the second identity.

$\square$

Corollary 4.2. *For a finite number of variables*

$$\prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - x_i y_j} = \sum_{\lambda \in \mathscr{P}} z_\lambda^{-1} p_\lambda(x_1, \ldots, x_n) p_\lambda(y_1, \ldots, y_m), \qquad (4.170)$$

and

$$\prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - x_i y_j} = \sum_{\lambda} h_\lambda(x_1, \ldots, x_n) m_\lambda(y_1, \ldots, y_m) \qquad (4.171)$$

$$= \sum_{\lambda} m_\lambda(x_1, \ldots, x_n) h_\lambda(y_1, \ldots, y_m), \qquad (4.172)$$

where the sum in Equation (4.171) is on all partitions of maximally m parts and the sum in Equation (4.172) is on all partitions of maximally n parts.

Proof. Setting $x_{n+1}, x_{n+2}, \ldots$ and $y_{m+1}, y_{m+2}, \ldots$ to zero in Equation (4.159), identity (4.170) is immediately deduced.

Doing the same in both parts of Equation (4.160) proves Equations (4.171) and (4.172). One should only note that m_λ, as a polynomial in a finite number of variables, vanishes if $l(\lambda)$ is greater than the number of variables. $\square$

Lemma 4.3. *Identities*

$$\prod_{i \geqslant 1} \prod_{j \geqslant 1} (1 + x_i y_j) = \sum_{\lambda \in \mathscr{P}} \varepsilon_\lambda z_\lambda^{-1} p_\lambda(\boldsymbol{x}) p_\lambda(\boldsymbol{y}), \qquad (4.173)$$

and

$$\prod_{i \geqslant 1} \prod_{j \geqslant 1} (1 + x_i y_j) = \sum_{\lambda \in \mathscr{P}} m_\lambda(\boldsymbol{x}) e_\lambda(\boldsymbol{y}) = \sum_{\lambda \in \mathscr{P}} e_\lambda(\boldsymbol{x}) m_\lambda(\boldsymbol{y}), \qquad (4.174)$$

hold.

Proof. First note that regarding ω as an involution on $\Lambda[x_1, x_2, \ldots]$ and using Equation (4.92) we have

$$\omega \left(\prod_{i \geqslant 1} \prod_{j \geqslant 1} \frac{1}{1 - x_i y_j} \right) = \omega \left(\prod_{j \geqslant 1} H_{\boldsymbol{x}}(y_j) \right)$$

$$= \prod_{j \geqslant 1} \omega \left(H_{\boldsymbol{x}}(y_j) \right). \qquad (4.175)$$

From this by using Equations (4.142) and (4.73) we get

$$\omega\left(\prod_{i\geqslant 1}\prod_{j\geqslant 1}\frac{1}{1-x_iy_j}\right) = \prod_{i\geqslant 1}\prod_{j\geqslant 1}(1+x_iy_j). \tag{4.176}$$

Of course, if ω was regarded as an involution on $\Lambda[y_1, y_2, \ldots]$, Equation (4.176) still holds.

To prove the identity (4.173), it suffices to act on both sides of Equation (4.159) by ω and make use of Equations (4.148) and (4.176).

For the second identitites, Applying ω as an involution on $\Lambda[x_1, x_2, \ldots]$ on both sides of the first part of Equation (4.160) and using Equations (4.136) and (4.176) gives rise to the first part of the second identity. Similarly, applying ω as an involution on $\Lambda[y_1, y_2, \ldots]$ on both sides of of the second part of Equation (4.160) and using the same Equation (4.136) gives rise to the second part of the second identity. $\qquad\square$

Theorem 4.4. *For a finite number of variables*

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(1+x_iy_j) = \sum_{\boldsymbol{\lambda}\in\mathscr{P}}\varepsilon_{\boldsymbol{\lambda}}z_{\boldsymbol{\lambda}}^{-1}p_{\boldsymbol{\lambda}}(x_1, \ldots, x_n)p_{\boldsymbol{\lambda}}(y_1, \ldots, y_m), \tag{4.177}$$

and

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(1+x_iy_j) = \sum_{\boldsymbol{\lambda}}m_{\boldsymbol{\lambda}}(x_1, \ldots, x_n)e_{\boldsymbol{\lambda}}(y_1, \ldots, y_m), \tag{4.178}$$

$$= \sum_{\boldsymbol{\lambda}}e_{\boldsymbol{\lambda}}(x_1, \ldots, x_n)m_{\boldsymbol{\lambda}}(y_1, \ldots, y_m), \tag{4.179}$$

where the sum in Equation (4.178) is on all partitions $\boldsymbol{\lambda}$ for which $l(\boldsymbol{\lambda}) \leqslant n$ and $\lambda_1 \leqslant m$, and the sum in Equation (4.179) is on all partitions $\boldsymbol{\lambda}$ for which $l(\boldsymbol{\lambda}) \leqslant m$ and $\lambda_1 \leqslant n$.

Proof. To prove these identities, it suffices to set x_{n+1}, x_{n+2}, $\ldots$ and y_{m+1}, y_{m+2}, $\ldots$ to zero in Lemma 4.3. For the second identities one should also note that $m_{\boldsymbol{\lambda}}$, as a polynomial in a finite number of variables, vanishes if and only if $l(\boldsymbol{\lambda})$ is greater than the number of variables and $e_{\boldsymbol{\lambda}}$, as such a polynomial, vanishes if and only if λ_1 is greater than the number of variables. $\qquad\square$

Decomposition of the Laughlin State and the Rank Saturation Conjecture

This chapter tries to find a weak Schmidt decomposition of the Laughlin state and delves more into the contents of Conjecture 1.1 and Conjecture 1.2, as outlined below.

In Section 5.1 we try to determine a weak Schmidt decomposition for a generic Laughlin state Ψ_m for a particle cut of the system. Typically this is a hard problem. To make progress we do a mathematical trick and define a transformation $\mathcal{C}_m$ and then using this transformation we reduce the problem to the simpler problem of determining a weak Schmidt decomposition of a "copy" of Ψ_1 but with larger number of variables. The latter problem is easier since we can use Theorem 4.4 to easily write this copy in a decomposed form. As is explained at the end of this section, by applying $\mathcal{C}_m$ on both sides of this decomposition, we find a decomposition of Ψ_m, which has the same mathematical form as of a weak Schmidt decomposition. The number of summands in this decomposition can be counted by Equation (4.42) and Theorem 3.3 then makes it possible to determine the rank of the reduced density operator. All this is applicable on the condition that this decomposition is in fact a weak Schmidt decomposition of Ψ_m. As we see at the end of this section, that this decomposition is actually a weak decomposition for Ψ_m translates itself to proving that the kernel of $\mathcal{C}_m$ is trivial. This motivates our study of properties of symmetric polynomials that vanish under $\mathcal{C}_m$, which is followed in Chapter 6.

In Section 5.2 we show that by a simple argument one can convince oneself that the rank of the reduced density operator is actually bounded from above.

In Section 5.3, we show that the upper bound for the reduced density operator found in Section 5.2 can be viewed as the number of quasi-hole states for a FQH system of reduced size, in the sense of the number of particles, but with an appropriate number of additional flux quanta. This chapter is based on the accompanied paper.

5.1   Weak Schmidt Decomposition of Ψ_m

Consider a FQH system $\mathcal{S}$ consisting of N electrons with coordinates z_1 till z_N in a pure state that can be modeled by a Laughlin state of filling factor $\nu = 1/m$,

$$\Psi_m(z_1, \ldots, z_N) = \prod_{1 \leqslant i < j \leqslant N}(z_i - z_j)^m \exp\Big(-\frac{1}{4l_B^2}\sum_{k=1}^{N}|z_k|^2 \Big), \qquad (5.1)$$

where m is an odd integer. As mentioned in Chapter 1, one way to probe the properties of the $\mathcal{S}$ is to look at the rank of the reduced density operator corresponding to a particle cut of the system $\mathcal{S}$. Consider the particle cut in which we declare the electrons numbered 1 till N_A to constitute subsystem A and electrons numbered $N_A + 1$ till N to constitute subsystem B. Here $N_B = N - N_A$ is the number of electrons in subsystem B and without the loss of generality one can assume that $N_A \leqslant N_B$. Let x_1 till x_{N_A} indicate the coordinates of particles in A and y_1 till y_{N_B} indicate the coordinates of particles in B. One should note that $\mathcal{H}^A$ is the space of all physically acceptable totally anti-symmetric functions in variables x_1 till x_{N_A} and $\mathcal{H}^B$ is the space of all physically acceptable totally anti-symmetric functions in variables y_1 till y_{N_B}. The goal is to try to determine the rank of the reduced density operator ρ^A corresponding to the pure state $\Psi_m(z_1, \ldots, z_N)$. To this end, by Theorem 3.3, it suffices to find a weak Schmidt decomposition of $\Psi_m(z_1, \ldots, z_N)$ and count the number of summands in that decomposition. Taking $\boldsymbol{x} = (x_1, \ldots, x_{N_A})$, $\boldsymbol{y} = (y_1, \ldots, y_{N_B})$, and $\boldsymbol{z} = (z_1, \ldots, z_N)$, Equation (5.1) can be written as

$$\Psi_m(\boldsymbol{z}) = F_m(\boldsymbol{x})\Phi_m(\boldsymbol{x}, \boldsymbol{y})G_m(\boldsymbol{y}), \qquad (5.2)$$

where

$$F_m(\boldsymbol{x}) = \prod_{i_1 < i_2}(x_{i_1} - x_{i_2})^m \exp\Big(-\frac{1}{4l_B^2}\sum_{k=1}^{N_A}|x_k|^2 \Big), \qquad (5.3)$$

$$\Phi_m(\boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{N_A}\prod_{j=1}^{N_B}(x_i - y_j)^m, \qquad (5.4)$$

$$G_m(\boldsymbol{y}) = \prod_{j_1 < j_2}(y_{j_1} - y_{j_2})^m \exp\Big(-\frac{1}{4l_B^2}\sum_{k=1}^{N_B}|y_k|^2 \Big). \qquad (5.5)$$

We now define the *clustering transformation* $\mathcal{C}_m$ as an algebra endomorphism from Λ_{mN_B} to Λ_{N_B} such that

$$\mathcal{C}_m(s) = t, \qquad (5.6)$$

where

$$t(y_1, y_2 \ldots, y_{N_B}) = s(\underbrace{y_1, \ldots, y_1}_{m}, \underbrace{y_2, \ldots, y_2}_{m}, \ldots, \underbrace{y_{N_B}, \ldots, y_{N_B}}_{m}). \qquad (5.7)$$

It is straight forward to see that, if $\boldsymbol{w} = (w_1, \ldots, w_{mN_B})$, the effect of clustering transformation $\mathcal{C}_m$ on

$$\Omega(\boldsymbol{x}, \boldsymbol{w}) := \prod_{i=1}^{N_A} \prod_{j=1}^{mN_B} (x_i - w_j), \tag{5.8}$$

considered as a symmetric polynomial in Λ_{mN_B} with w variables, is as follows:

$$\mathcal{C}_m(\Omega)(\boldsymbol{x}, \boldsymbol{w}) = \Phi_m(\boldsymbol{x}, \boldsymbol{y}). \tag{5.9}$$

On the other hand, pulling out the x's in Equation (5.8) yields

$$\Omega(\boldsymbol{x}, \boldsymbol{w}) = \left[\prod_{i=1}^{N_A} x_i^{mN_B} \right] \left[\prod_{i=1}^{N_A} \prod_{j=1}^{mN_B} \left(1 - \frac{1}{x_i} w_j \right) \right]$$

$$= \left[\prod_{i=1}^{N_A} x_i^{mN_B} \right] \sum_{\boldsymbol{\lambda}} m_{\boldsymbol{\lambda}}(-1/\boldsymbol{x}) e_{\boldsymbol{\lambda}}(\boldsymbol{w}), \tag{5.10}$$

where $(-1/\boldsymbol{x})$ is a shorthand for $(-1/x_1, \ldots, -1/x_{N_A})$. In the last step, the first part of Equation (4.178) is used and, therefore, the sum is over all partitions that fit into a rectangle of height N_A and width mN_B. So, from Equation (4.42), this sum consists of

$$l_m = \binom{N_A + mN_B}{N_A} \tag{5.11}$$

number of terms. It is straight forward to check that

$$\left[\prod_{i=1}^{N_A} x_i^{mN_B} \right] m_{\boldsymbol{\lambda}}(-1/\boldsymbol{x}) = (-1)^{|\boldsymbol{\lambda}|} m_{\bar{\boldsymbol{\lambda}}}(\boldsymbol{x}), \tag{5.12}$$

where $\bar{\boldsymbol{\lambda}}$ is the complement of $\boldsymbol{\lambda}$ with respect to the rectangle of height N_A and width mN_B defined by Equation (4.47). Hence, Equation (5.10) can be written as

$$\Omega(\boldsymbol{x}, \boldsymbol{w}) = \sum_{\boldsymbol{\lambda}} (-1)^{|\boldsymbol{\lambda}|} m_{\bar{\boldsymbol{\lambda}}}(\boldsymbol{x}) e_{\boldsymbol{\lambda}}(\boldsymbol{w}), \tag{5.13}$$

with l_m number of summands. Thus, from Equation (5.9),

$$\Phi_m(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{\lambda}} (-1)^{|\boldsymbol{\lambda}|} m_{\bar{\boldsymbol{\lambda}}}(\boldsymbol{x}) \mathcal{C}_m(e_{\boldsymbol{\lambda}})(\boldsymbol{y}). \tag{5.14}$$

Plugging this back into Equation (5.2), yields

$$\Psi_m(\boldsymbol{z}) = \sum_{\boldsymbol{\lambda}} \left(F_m(\boldsymbol{x})(-1)^{|\boldsymbol{\lambda}|} m_{\bar{\boldsymbol{\lambda}}}(\boldsymbol{x}) \right) \left(\mathcal{C}_m(e_{\boldsymbol{\lambda}})(\boldsymbol{y}) G_m(\boldsymbol{y}) \right). \tag{5.15}$$

This is a weak Schmidt decomposition of $\Psi_m(\boldsymbol{z})$ provided that the set

$$\left\{ \mathcal{C}_m(e_{\boldsymbol{\lambda}}) \mid \lambda_1 \leqslant mN_B , \, l(\boldsymbol{\lambda}) \leqslant N_A \right\} \tag{5.16}$$

forms a linearly independent subset of Λ_{N_B}. Therefore proving that the rank of the reduced density operator ρ^A, when the total system $\mathcal{S}$ is in the pure state $\Psi_m(\boldsymbol{z})$, is given by l_m in Equation (5.11), boils down to proving that the set (5.16) is linearly independent. On the other hand, as we see in (4.84), the set

$$\left\{ e_{\boldsymbol{\lambda}}(w_1, \ldots, w_{mN_B}) \mid \lambda_1 \leqslant mN_B , \, l(\boldsymbol{\lambda}) \leqslant N_A \right\} \tag{5.17}$$

is a basis for $\Lambda_{mN_B}^{N_A}$ and therefore it is linearly independent. Thus if one can show that the restriction of $\mathcal{C}_m$ to $\Lambda_{mN_B}^{N_A}$, which is a linear map to $\Lambda_{N_B}^{mN_A}$, is injective as long as $N_A \leqslant N_B$, it proves that the set (5.16) is linearly independent, since injective linear maps respect linear independence. Moreover, a linear map is injective if and only if its kernel is trivial. Thus one needs to show that, besides the zero polynomial, no symmetric polynomial in mN_B variables and maximum degree N_A can vanish under $\mathcal{C}_m$. This, in fact, is the content of Conjecture 1.2 that motivates the subject on the next chapter.

It is worth noting that for $m = 1$, the clustering transformation $\mathcal{C}_1$ is just the identity map on Λ_{N_B} and, therefore, the set (5.16) reduces to

$$\left\{ e_{\boldsymbol{\lambda}}(y_1, \ldots, y_{N_B}) \mid \lambda_1 \leqslant N_B , \, l(\boldsymbol{\lambda}) \leqslant N_A \right\}. \tag{5.18}$$

Comparing this with the set (4.84) makes it clear that this set is a basis for $\Lambda_{N_B}^{N_A}$ and, consequently, linearly independent. Thus the rank of the reduced density operator in this simple case is indeed

$$l_1 = \binom{N_A + N_B}{N_A}. \tag{5.19}$$

5.2 Upper Bound for the Rank of the Reduced Density Operator

In this section, by a straight forward argument, we show that l_m given in Equation (5.11), is an upper bound of the reduced density operator for the Laughlin state (5.1) when the system is subjected to a particle cut as described on the first paragraph of the last section. Consider a Schmidt decomposition of $\Psi_m(\boldsymbol{z})$ as follows[§]:

$$\Psi_m(\boldsymbol{z}) = \sum_{i=1}^{r} \lambda_i \, \phi_i^A(\boldsymbol{x}) \phi_i^A(\boldsymbol{y}). \tag{5.20}$$

[§]In the following argument the Gaussian exponential term of the Laughlin wave function has no effect and it is suppressed.

Since $\Psi_m(\boldsymbol{z})$ vanishes as mth power when two variables coincide, this implies that $\phi_i^A(\boldsymbol{x})$ vanishes as mth power when two x coordinates coincide. Therefore,

$$\phi_i^A(\boldsymbol{x}) = P_i^A(\boldsymbol{x}) \prod_{1 \leqslant i < j \leqslant N_A} (x_i - x_j)^m, \tag{5.21}$$

where $P_i^A(\boldsymbol{x})$ is a symmetric polynomial in N_A variables of some degree d. Since the degree of the symmetric polynomial expressed as a product on the right hand side of Equation (5.21) is $m(N_A - 1)$ and the degree of Ψ_m is $m(N - 1)$, one should have

$$d + m(N_A - 1) \leqslant m(N - 1), \tag{5.22}$$

or

$$d \leqslant mN_B. \tag{5.23}$$

Hence, the polynomials $P_i^A(\boldsymbol{x})$ belong to $\Lambda_{N_A}^{mN_B}$ and by Equation (4.62) the number of linearly independent such polynomials is at most l_m and, consequently, the number of linearly independent $\phi_i^A(\boldsymbol{x})$'s, which is an upper bound for the rank of the reduced density operator of Ψ_m, is also at most l_m. It shown in section 5.3 that l_m is precisely equal to the number of quasi-hole states for a system of only N_A particles, but with mN_B additional flux quanta.

One should note that if one could prove Conjecture 1.2, then this would mean that the rank of the reduced density operator of a FQH system modeled by the Laughlin state and subjected to a particle cut always reaches this upper bound. In other words the rank of the reduced density operator is "saturated".

5.3 Rank Saturation Conjecture

In this section we expand the content of Conjecture 1.1 and then determine the number of quasi-hole states considering the Laughlin state as the model state.

Consider a FQH system $\mathcal{S}$ consisting of $N = N_A + N_B$ electrons with $N_A \leqslant N_B$ that is described by a model state, like Laughlin or Moore–Read state, at a generic filling factor ν. As mentioned in Chapter 2, these model states are exact ground states of some model Hamiltonians.

Any ground state of the corresponding model Hamiltonian is called a *quasi-hole state*. It was also mentioned in Chapter 2, for the special case of the Laughlin state, that quasi-hole states are suitable trial wave functions to describe the system $\mathcal{S}$ with some number of flux quanta added locally to the system. These states are usually referred to as quasi-hole excitations of the corresponding model state. In this context, the model state itself can be regarded as a quasi-hole state corresponding to zero number of additional flux quanta or quasi-holes. The number of quasi-hole states $\#\mathrm{q.h.}(N, N_\Phi)$ for a

FQH system with N number of electrons and N_Φ number of flux quanta can often be obtained exactly [RR96, GR01, ARRS01, Ard02, Rea06].

Now let us open the content of Conjecture 1.1. Suppose that the system $\mathcal{S}$ described above is subjected to a particle cut that divides $\mathcal{S}$ into two subsystems A and B with N_A and N_B number of electrons, respectively, and let $r_\nu^A(N_A, N_B)$ be the rank of the reduced density operator ρ^A. Conjecture 1.1 claims that

$$r_\nu^A(N_A, N_B) = \#\text{q.h.}(N_A, \nu^{-1}N_B). \tag{5.24}$$

Now let us consider the special case of the Laughlin state Ψ_m at fill factor $\nu = 1/m$, which is our interest in this thesis, and calculate the right hand side of Equation (5.24).

Consider a FQH system consisting of N_A number of particles and $\nu^{-1}N_B = mN_B$ number of flux quanta located at unspecified points with complex coordinates w_1 till w_{mN_B}. It turns out that the ground state $\Psi_m^{\text{q.h.}}$ of the corresponding model Hamiltonian takes the following form:

$$\Psi_m^{\text{q.h.}}(z_1, \ldots, z_{N_A}) = \Psi_m(z_1, \ldots, z_{N_A})P_{\boldsymbol{w}}(z_1, \ldots, z_{N_A}), \tag{5.25}$$

where Ψ_m is the Laughlin wave function at filling factor $1/m$, $P_{\boldsymbol{w}}(z_1, \ldots, z_{N_A})$, with $\boldsymbol{w} = (w_1, \ldots, w_{mN_B})$ being the coordinates of the added flux quanta, is a symmetric polynomial in z's. On the other hand, the degree of Ψ_m is $m(N_A-1)$ and the degree of $\Psi_m^{\text{q.h.}}$ is $m(N_A-1)+mN_B$. Thus $P_{\boldsymbol{w}}(z_1, \ldots, z_{N_A})$ is a symmetric polynomial in N_A variables and the degree at most mN_B, that is, $P_{\boldsymbol{w}}(z_1, \ldots, z_{N_A})$ belongs to $\Lambda_{N_A}^{mN_B}$. Therefore, by Equation (4.62), for the special case of the Laughlin state Ψ_m

$$\#\text{q.h.}(N_A, \nu^{-1}N_B) = \binom{N_A + mN_B}{N_A}. \tag{5.26}$$

This is the same upper bound found for the rank of the reduced density operator in the last section.

One should note that for the special case of $m = 1$, Equation (5.24) claims

$$r_1^A(N_A, N_B) = \#\text{q.h.}(N_A, N_B), \tag{5.27}$$

which is true based on Equations (5.19) and (5.26). That is the rank saturation conjecture for the Laughlin state Ψ_1 is seen to be correct in this simple case.

Clustering Properties of Symmetric Polynomials

The goal of this chapter is to explore some characteristics of symmetric polynomials that vanish under the clustering transformation $\mathcal{C}_m$ introduced in the last chapter by Equations (5.6) and (5.7). In this chapter, however, we suppress the subscript B and simply write N instead of N_B. Moreover, in what follows we assume that m and N are given positive integers.

In Section 6.1 we prove that a necessary condition for a non-zero symmertic polynomial in mN variables to vanish under $\mathcal{C}_m$ is that its *total* degree be at least $N + 1$. It is also shown that there is a unique (up to a scaling numerical factor) symmetric polynomial of total degree $N + 1$ in mN variables that vanishes under $\mathcal{C}_m$. We also present a constructive way to determine these symmetric polynomials with the mentioned propety and introduce a basis for the ideal of Λ_{mN} consisting of symmetric polynomials that vanish under $\mathcal{C}_m$. All is realized by introducing a new family of generators for Λ_{mN} that are engineered so that they behave in a controllable manner under $\mathcal{C}_m$.

Section 6.2, which is divided into five subsections, is dedicated to exploring some properties of this new family of generators that might pave the way for more investigations in the future. This chapter is based on the accompanied paper [GEA15].

6.1   New Generating Set for Λ_{mN}

From all algebraically independent sets of generators introduced for Λ_{mN} in Chapter 4 only the elements of the set $\{\, p_1, \ldots, p_{mN} \,\}$, consisting of power-sum symmetric polynomials p_i, have a simple behavior under $\mathcal{C}_m$ that is

$$\mathcal{C}_m(p_i)(w_1, \ldots, w_{mN}) = m\, p_i(y_1, \ldots, y_N). \qquad (6.1)$$

Despite of this simple behavior, these polynomials don't seem convenient if one needs to describe the property of polynomials in Λ_{mN} that vanish when acted on by $\mathcal{C}_m$, so we try to explore a new set of generators for Λ_{mN} that are suitable in this regard. The strategy is to introduce a family of polynomials in Λ_{mN}, which depend on a real parameter and we show that any member of

this family corresponding to a non-zero value of the parameter constitutes an algebraically independent generating set for Λ_{mN}.

Let x be a real parameter and let n be a non-negative integer. The polynomial $r_n^{(x)}$ in Λ_{mN} is defined by

$$r_n^{(x)} = n! \sum_{\boldsymbol{\lambda} \vdash n} (-x)^{l(\boldsymbol{\lambda})} \frac{p_{\boldsymbol{\lambda}}}{z_{\boldsymbol{\lambda}}}, \tag{6.2}$$

where $z_{\boldsymbol{\lambda}}$ is defined by Equation (4.155). It is clear that $r_n^{(x)}$ is a symmetric polynomial of total degree n. In Section 6.2.1 it is shown that the degree of this polynomial is also n. The first five of these polynomials are listed below.

$$r_0^{(x)} = 1, \tag{6.3}$$
$$r_1^{(x)} = -xp_1, \tag{6.4}$$
$$r_2^{(x)} = x^2 p_1^2 - xp_2, \tag{6.5}$$
$$r_3^{(x)} = -x^3 p_1^3 + 3x^2 p_1 p_2 - 2xp_3, \tag{6.6}$$
$$r_4^{(x)} = x^4 p_1^4 - 6x^3 p_1^2 p_2 + 3x^2 p_2^2 + 8x^2 p_1 p_3 - 6xp_4. \tag{6.7}$$

From Equation (6.2), it is clear that for a given positive integer n, the term corresponding to $\boldsymbol{\lambda} = (n)$ in $r_n^{(x)}$ is a monomial in x times p_n. Therefore, iterative computation shows that, for a non-zero value of the parameter x and any positive integer n, the power-sum p_n can be written as a polynomial in $r_1^{(x)}$ till $r_n^{(x)}$ with functions of x as coefficients. Thus, since $\{p_1, \ldots, p_{mN}\}$ is an algebraically independent generating set for Λ_{mN}, so is $\{r_1^{(x)}, \ldots, r_{mN}^{(x)}\}$ for any non-zero x. Therefore we proved the following lemma:

Lemma 6.1. *For any non-zero value of the parameter x, the set $\{r_1^{(x)}, \ldots, r_{mN}^{(x)}\}$ forms an algebraically independent generating set for Λ_{mN}.*

The key property of polynomials $r_n^{(x)}$ is that they behave nicely under the action of $\mathcal{C}_m$, namely

$$\mathcal{C}_m\big(r_n^{(x)}\big) = r_n^{(mx)}. \tag{6.8}$$

This is readily seen by applying $\mathcal{C}_m$ on both sides of Equation (6.2) and using

$$\mathcal{C}_m(p_{\boldsymbol{\lambda}}) = m^{l(\boldsymbol{\lambda})} p_{\boldsymbol{\lambda}}, \tag{6.9}$$

which, in turn, is a consequence of Equation (6.1). Replacing x by -1 in Equation (6.2) and using Equation (4.156), yields

$$r_n^{(-1)} = n! \, h_n, \tag{6.10}$$

and similarly, replacing x by 1 in that equation and making use of Equation (4.158), yields

$$r_n^{(1)} = (-1)^n n! \, e_n. \tag{6.11}$$

Now, for a non-negative integer n, we define the nth *modified power-sum* symmetric polynomial $\widetilde{p}_n$ as an element in Λ_{mN} by

$$\widetilde{p}_n = \frac{(-1)^n}{n!} r_n^{(1/m)}. \tag{6.12}$$

The total degree of $\widetilde{p}_n$ is n and by Lemma 6.1, the set $\left\{\widetilde{p}_1, \ldots, \widetilde{p}_{mN}\right\}$ is an algebraically independent generating set for Λ_{mN}. Actually, these polynomials are engineered to enjoy the property

$$\mathcal{C}_m(\widetilde{p}_n) = e_n, \tag{6.13}$$

that can be obtained by applying $\mathcal{C}_m$ on both sides of Equation (6.12) and using Equations (6.8) and (6.11). We remind the reader that $\widetilde{p}_n$ in Equation (6.13) contains mN variables while e_n in that equation contains N variables. Hence, we have the following lemma:

Lemma 6.2. *The set $\mathcal{G} = \left\{\widetilde{p}_1, \ldots, \widetilde{p}_{mN}\right\}$ is an algebraically independent generating set for Λ_{mN} and $\mathcal{C}_m(\widetilde{p}_n) = e_n$ for all integers n, $1 \leqslant n \leqslant mN$.*

Based on the fact that $e_n = 0$ for $n > N$ and Lemma 6.2, we can conclude that:

Corollary 6.3. *For $N + 1 \leqslant n \leqslant mN$, $\mathcal{C}_m(\widetilde{p}_n) = 0$.*

Polynomials $\widetilde{p}_n$'s have the exact right property that enables us to prove the following theorem.

Theorem 6.4. *A non-zero symmetric polynomial in Λ_{mN} vanishes under $\mathcal{C}_m$ if and only if, when expressed as a polynomial in $\widetilde{p}_1$ till $\widetilde{p}_{mN}$, each monomial term contains some power of at least one of the polynomials $\widetilde{p}_{N+1}$ till $\widetilde{p}_{mN}$.*

Proof. First, Let s be a symmetric polynomial in Λ_{mN} such that $\mathcal{C}_m(s) = o$. By Lemma 6.1, there exists a polynomial r in mN variables such that

$$s(w_1, \ldots, w_{mN}) = r(\widetilde{p}_1, \ldots, \widetilde{p}_{mN}). \tag{6.14}$$

In general, there are two kinds of monomials on the right hand side of the equation above, those that contain some power of at least one of the polynomials $\widetilde{p}_{N+1}$ till $\widetilde{p}_{mN}$ and those that don't. Thus, $s(w_1, \ldots, w_{mN})$ can be decomposed uniquely into two parts as follows

$$s(w_1, \ldots, w_{mN}) = a(\widetilde{p}_1, \ldots, \widetilde{p}_N) + b(\widetilde{p}_1, \ldots, \widetilde{p}_{mN}), \tag{6.15}$$

where $a(\widetilde{p}_1, \ldots, \widetilde{p}_N)$ consists of the monomials on the right hand side of Equation (6.14) that do not depend on any of the polynomials $\widetilde{p}_{N+1}$ till $\widetilde{p}_{mN}$ and $b(\widetilde{p}_1, \ldots, \widetilde{p}_{mN})$ consists of the rest. Based on its construction and Corollary 6.3, $b(\widetilde{p}_1, \ldots, \widetilde{p}_{mN})$ vanishes under $\mathcal{C}_m$.

Applying $\mathcal{C}_m$ on both sides of Equation (6.15) and making use of Equation (6.13) yields

$$\mathcal{C}_m(s)(y_1, \ldots, y_N) = a(e_1, \ldots, e_N), \tag{6.16}$$

and therefore $a(e_1, \ldots, e_N)$ vanishes. From Chapter 4 we know that the set $\{e_1, \ldots, e_N\}$ as polynomials in Λ_N are algebraically independent. Therefore, a is the zero polynomial and from Equation (6.15) we get

$$s(w_1, \ldots, w_{mN}) = b(\widetilde{p}_1, \ldots, \widetilde{p}_{mN}). \tag{6.17}$$

The converse is obviously true. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Since the total degree of $\widetilde{p}_n$ is equal to n for all positive integers n, the following two corollaries are immediate consequences of Theorem 6.4.

Corollary 6.5. *In Λ_{mN}, the polynomial $\widetilde{p}_{N+1}$ is the unique (up to an overall factor) symmetric polynomial of total degree $N+1$ that vanishes under $\mathcal{C}_m$.*

Corollary 6.6. *In Λ_{mN}, there is no non-zero symmetric polynomial with total degree less than $N+1$ that vanishes under $\mathcal{C}_m$.*

In other words the kernel of the restriction of $\mathcal{C}_m$ to $\Lambda_{mN,N}$ is trivial. At this stage it might be instructive to state a brief version of Conjecture 1.2 here again to compare it with the statement in Corollary 6.6.

Conjecture 1.2. *In Λ_{mN}, there is no non-zero symmetric polynomial with degree less than $N+1$ that vanishes under $\mathcal{C}_m$.*

To formulate Theorem 6.4 in the language of Chapter 4, for a given partition $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$, we define $\widetilde{p}_{\boldsymbol{\lambda}}$ by

$$\widetilde{p}_{\boldsymbol{\lambda}} = \widetilde{p}_{\lambda_1} \cdots \widetilde{p}_{\lambda_r}. \tag{6.18}$$

Thus, Theorem 6.4 asserts that a symmetric polynomial s in Λ_{mN} of total degree D vanishes under the clustering transformation $\mathcal{C}_m$ if and only if

$$s = \sum_{\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}} \widetilde{p}_{\boldsymbol{\lambda}}, \tag{6.19}$$

where $a_{\boldsymbol{\lambda}}$'s are rational numbers and the sum is over all partitions $\boldsymbol{\lambda}$ of D such that $\lambda_1 \leqslant mN$ and the multiplicity of at least one of the numbers $N+1$ till mN in $\boldsymbol{\lambda}$ is non-zero. This is equivalent to saying that $N+1 \leqslant \lambda_1 \leqslant mN$. Hence:

Corollary 6.7. *The set*

$$\left\{ \widetilde{p}_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \vdash D , \, N+1 \leqslant \lambda_1 \leqslant mN \right\}, \tag{6.20}$$

is a basis for the vector subspace of $\Lambda_{mN,D}$ consisting of symmetric polynomials that vanish under the clustering transformation $\mathcal{C}_m$ and the set

$$\left\{ \widetilde{p}_{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathscr{P} , \, N + 1 \leqslant \lambda_1 \leqslant mN \right\}, \tag{6.21}$$

is a basis for the ideal of Λ_{mN} consisting of symmetric polynomials that vanish under the clustering transformation $\mathcal{C}_m$.

Example 6.1. Consider the simplest non-trivial case where $m = 2$ and $N = 1$. We want to describe all non-zero polynomials in Λ_2 that vanish under clustering transformation $\mathcal{C}_2$. Assume that s is such a polynomial. As in Equation (6.17),

$$s(w_1, w_2) = b\big(\widetilde{p}_1(w_1, w_2), \widetilde{p}_2(w_1, w_2)\big), \tag{6.22}$$

for some polynomial b that each term in $b(\widetilde{p}_1, \widetilde{p}_2)$ contains some power of $\widetilde{p}_2$. From Equations (6.5) and (6.12),

$$\begin{aligned}
\widetilde{p}_2(w_1, w_2) &= \frac{1}{8} \left(p_1^2(w_1, w_2) - 2p_2(w_1, w_2) \right) \\
&= \frac{1}{8}(w_1 - w_2)^2.
\end{aligned} \tag{6.23}$$

Thus

$$s(w_1, w_2) = (w_1 - w_2)^2 q(w_1, w_2), \tag{6.24}$$

where q is some symmetric polynomial in Λ_2.

Example 6.2. As another example, consider the case in which $m = N = 2$. Let s be a symmetric polynomial in Λ_4 such that $\mathcal{C}_2(s) = o$. As in the previous example,

$$s = b(\widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3, \widetilde{p}_4). \tag{6.25}$$

where b is a polynomial such that each term in $b(\widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3, \widetilde{p}_4)$ contains at least some power of $\widetilde{p}_3$ or $\widetilde{p}_4$ (or both). So

$$s = \widetilde{p}_3 q_1 + \widetilde{p}_4 q_2 + \widetilde{p}_3 \widetilde{p}_4 q_3, \tag{6.26}$$

where q_1, q_2, and q_3 are some symmetric polynomials in Λ_4 such that q_1 does not involve $\widetilde{p}_4$ and q_2 does not involve $\widetilde{p}_3$. Moreover

$$\widetilde{p}_3 = \frac{1}{48}(p_1^3 - 6p_2 p_1 + 8p_3), \tag{6.27}$$

and

$$\widetilde{p}_4 = \frac{1}{384}(p_1^4 - 12p_1^2 p_2 + 12p_2^2 + 32p_1 p_3 - 48p_4), \tag{6.28}$$

each in four variables.

Before ending this section some comment on the Conjecture 1.2, rephrased as the one on page 60, seems in order. Despite the apparent similarity between statements of the Corollary 6.6 and the Conjecture 1.2, it turned out that the proof for Conjecture 1.2 is much harder and highly non-trivial. This is due to the fact that upon taking linear combinations of symmetric polynomials the *total* degree does not change, as long as the resulting polynomial does not vanish while the *degree* can be lowered.

6.2   Properties of Polynomials in $\mathcal{G}$

This section is devoted to probing some properties of the polynomials $\widetilde{p}_i$ in $\mathcal{G}$ introduced in the last section. This includes, determining the degree, the generating function, the determinant expansion, the monomial decomposition and finally how they behave under translation. Since $\widetilde{p}_n$ and $r_n^{(x)}$ differ only by a multiplicative factor, it suffices to go through these steps only for the latter.

6.2.1   Degree of $r_n^{(x)}$

Consider the monomial decomposition of $r_n^{(x)}$

$$r_n^{(x)} = \sum_{\boldsymbol{\lambda} \vdash n} q_n^{(\boldsymbol{\lambda})}(x)\, m_{\boldsymbol{\lambda}}, \tag{6.29}$$

for a certain number of variables. It is clear that for those values of x that $q_n^{(\boldsymbol{\lambda}=(n))}(x)$, the coefficient of $m_{(n)}$, is non-zero the degree of $r_n^{(x)}$ is equal to n. This motivates us to search for those x such that this coefficient vanishes. From Equation (6.2) we get

$$r_n^{(x)}(w, 0, 0, \ldots, 0) = n! w^n \sum_{\boldsymbol{\lambda} \vdash n} \frac{(-x)^{l(\boldsymbol{\lambda})}}{z_{\boldsymbol{\lambda}}}, \tag{6.30}$$

and from Equation (6.29) we get

$$r_n^{(x)}(w, 0, 0, \ldots, 0) = q_n^{(\boldsymbol{\lambda}=(n))}(x)\, w^n. \tag{6.31}$$

Consequently

$$q_n^{(\boldsymbol{\lambda}=(n))}(x) = n! \sum_{\boldsymbol{\lambda} \vdash n} \frac{(-x)^{l(\boldsymbol{\lambda})}}{z_{\boldsymbol{\lambda}}}, \tag{6.32}$$

that is, $q_n^{(\boldsymbol{\lambda}=(n))}(x)$ is a polynomial in x of total degree n. Now we show that n integers $0, 1, \ldots, n-1$ are the roots of $q_n^{(\boldsymbol{\lambda}=(n))}(x)$. Let m be any of these integers. Then using Equations (6.8) and (6.11),

$$r_n^{(m)} = \mathcal{C}_m\big(r_n^{(1)}\big) = (-1)^n n!\, \mathcal{C}_m(e_n). \tag{6.33}$$

Hence,

$$r_n^{(m)}(w, 0, 0, \ldots, 0) = (-1)^n n! \, e_n(\underbrace{w, \ldots, w}_{m}, 0, \ldots, 0)$$

$$= (-1)^n n! \, e_n(\underbrace{w, \ldots, w}_{m})$$

$$= 0. \tag{6.34}$$

The last step is because the number of variables is less than n. So Equation (6.31) implies that $q_n^{(\boldsymbol{\lambda}=(n))}(m) = 0$. Thus if $x \neq 0, 1, 2, \ldots, n-1$, then the degree of $r_n^{(x)}$, as its total degree, is equal to n.

From Equation (6.32), it is also clear that for large values of x,

$$q_n^{(\boldsymbol{\lambda}=(n))}(x) \sim (-1)^n x^n \tag{6.35}$$

and consequently,

$$q_n^{(\boldsymbol{\lambda}=(n))}(x) = (-1)^n x(x-1)(x-2) \cdots (x-n+1). \tag{6.36}$$

An explicit expression for $q_n^{(\boldsymbol{\lambda})}(x)$ for all partitions $\boldsymbol{\lambda}$ is given in Section 6.2.4.

6.2.2   Generating Function for $r_n^{(x)}$

The polynomials $r_n^{(x)}$ fulfill the following generating function formula

$$\exp\left(-x \sum_{k \geqslant 1} p_k \frac{q^k}{k} \right) = \sum_{n \geqslant 0} r_n^{(x)} \frac{q^n}{n!}. \tag{6.37}$$

To see this, one needs to Taylor expand formally the left hand side of the equation above. Then one should show that $n!$ times the coefficient of q^n in this expansion is $r_n^{(x)}$. A straight forward calculation shows that this coefficient equals

$$\frac{(-x)}{1!} \frac{p_n}{n} + \frac{(-x)^2}{2!} \sum_{\substack{r_1, r_2 \geqslant 1 \\ r_1 + r_2 = n}} \frac{p_{r_1} p_{r_2}}{r_1 \, r_2} + \cdots + \frac{(-x)^n}{n!} \sum_{\substack{r_1, \ldots, r_n \geqslant 1 \\ r_1 + \cdots + r_n = n}} \frac{p_{r_1} \cdots p_{r_n}}{r_1 \cdots r_n}. \tag{6.38}$$

A generic term in the sum above can be written as

$$\sum_{\substack{r_1, \ldots, r_k \geqslant 1 \\ r_1 + \cdots + r_k = n}} \frac{p_{r_1} \cdots p_{r_k}}{r_1 \cdots r_k} = \sum_{\substack{\boldsymbol{\lambda} \vdash n \\ l(\boldsymbol{\lambda}) = k}} \frac{k!}{\prod_{i \geqslant 1} m_i(\boldsymbol{\lambda})!} \frac{p_{\boldsymbol{\lambda}}}{\prod_{i \geqslant 1} i^{m_i(\boldsymbol{\lambda})}}$$

$$= k! \sum_{\substack{\boldsymbol{\lambda} \vdash n \\ l(\boldsymbol{\lambda}) = k}} \frac{p_{\boldsymbol{\lambda}}}{z_{\boldsymbol{\lambda}}}. \tag{6.39}$$

Consequently, $n!$ times the expression (6.38) is the right hand side of Equation (6.2).

It is worth to know that using Equation (6.37) one can prove the following identity:

$$r_n^{(x+y)} = \sum_{k=0}^{n} \binom{n}{k} r_k^{(x)} r_{n-k}^{(y)}. \tag{6.40}$$

6.2.3 Determinant Expansion for $r_n^{(x)}$

Acting on both sides of Equation (6.11) by $\mathcal{C}_m$, using Equations (6.8), (6.1), and (4.116), and finally relabelling m by x, give rise to

$$r_n^{(x)} = (-1)^n \begin{vmatrix} xp_1 & 1 & 0 & \dots & 0 \\ xp_2 & xp_1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ xp_{n-1} & xp_{n-2} & \dots & xp_1 & n-1 \\ xp_n & xp_{n-1} & \dots & xp_2 & xp_1 \end{vmatrix}. \tag{6.41}$$

6.2.4 Monomial Decomposition of $r_n^{(x)}$

In this section an explicit expression for $q_n^{(\lambda)}(x)$ is found that enables one, using Equation (6.29), to write the full monomial decomposition of $r_n^{(x)}$.

A direct application of the differentiation operator ∂_x on both sides of Equation (6.2) shows that

$$\left. \partial_x r_n^{(x)} \right|_{x=0} = -(n-1)!\, p_n, \tag{6.42}$$

for a positive integer n.

Lemma 6.8. *Let i be a given non-negative integer. For any non-negative integers j and n, the monomial expansion of $\left. \partial_x^j r_n^{(x)} \right|_{x=i}$ involves only partitions $\boldsymbol{\lambda}$ with $\lambda_{j+1} \leqslant i$.*

Proof. The proof is by induction on i for a fixed value of j. For $i = 0$, by a straight forward calculation on Equation (6.37), it is readily seen that for a given j and n,

$$\left. \partial_x^j r_n^{(x)} \right|_{x=0} = (-1)^j n! \sum_{\substack{k_1,\dots,k_j \geqslant 1 \\ k_1+\dots+k_j=n}} \prod_{r \geqslant 1} \frac{p_{k_r}}{k_r}. \tag{6.43}$$

Here each expression $\prod_r p_{k_r}$ has a monomial decomposition that involves only those partitions $\boldsymbol{\lambda}$ such that $l(\boldsymbol{\lambda}) \leqslant j$. Therefore, $\lambda_{j+1} \leqslant 0$ or, in fact, $\lambda_{j+1} = 0$.

Now assume that the statement is correct for $i - 1$. From Equation (6.40) we get,

$$\partial_x^j r_n^{(x+y)} = \sum_{k=0}^{n} \binom{k}{n} \partial_x^j r_k^{(x)} r_{n-k}^{(y)}. \tag{6.44}$$

Choosing $x = i - 1$ and $y = 1$ in equation above and using Equation (6.11) yields

$$\partial_x^j r_n^{(x)} \Big|_{x=i} = n! \sum_{k=0}^{n} \frac{(-1)^{n-k}}{k!} m_{(1^{n-k})} \left(\partial_x^j r_k^{(x)} \Big|_{x=i-1} \right). \tag{6.45}$$

Therefore, by induction hypothesis and the following formula

$$m_{\boldsymbol{\mu}} m_{\boldsymbol{\lambda}} = m_{\boldsymbol{\mu}+\boldsymbol{\lambda}} + \sum_{\boldsymbol{\nu} \prec \boldsymbol{\mu}+\boldsymbol{\lambda}} a_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\boldsymbol{\nu}} m_{\boldsymbol{\nu}}, \tag{6.46}$$

in which $a_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\boldsymbol{\nu}}$'s are non-negative integers and $\boldsymbol{\mu}+\boldsymbol{\lambda}$ is defined componentwise, the statement is proved for i. $\qquad \square$

Theorem 6.9. *The coefficients $q_n^{(\boldsymbol{\lambda})}(x)$ in the monomial expansion of $r_n^{(x)}$, where n is not less than the number of variables, is determined by the following equation*

$$q_n^{(\boldsymbol{\lambda})}(x) = (-1)^n \binom{n}{\lambda_1, \ldots, \lambda_n} \prod_{i=0}^{n-1} (x - i)^{\lambda'_{i+1}}, \tag{6.47}$$

where

$$\binom{n}{\lambda_1, \ldots, \lambda_n} = \frac{n!}{\lambda_1! \cdots \lambda_n!}, \tag{6.48}$$

and λ'_{i+1} is the $(i+1)$th component of the conjugate partition $\boldsymbol{\lambda}'$.

Proof. Let i and j be two non-negative integers. Equation (6.29) yields

$$\partial_x^j r_n^{(x)} \Big|_{x=i} = \sum_{\boldsymbol{\lambda} \vdash n} \left(\partial_x^j q_n^{(\boldsymbol{\lambda})}(x) \Big|_{x=i} \right) m_{\boldsymbol{\lambda}}. \tag{6.49}$$

Therefore Lemma 6.8 implies that

$$\partial_x^j q_n^{(\boldsymbol{\lambda})}(x) \Big|_{x=i} = 0, \tag{6.50}$$

for any partition $\boldsymbol{\lambda}$ of n such that $\lambda_{j+1} \geqslant i + 1$.

Now the claim is that for a given partition $\boldsymbol{\lambda}$ of n and any non-negative integer i, Equation (6.50) is satisfied only for $j = 0, 1, 2, \ldots, \lambda'_{i+1} - 1$. Let k be an integer such that $0 \leqslant k \leqslant (\lambda'_{i+1} - 1)$, then $\lambda_{k+1} \geqslant \lambda_{\lambda'_{i+1}} \geqslant i + 1$, since λ'_{i+1} is the number of parts of $\boldsymbol{\lambda}$ that are greater than or equal to $i + 1$. So Equation (6.50) is fulfilled for these values of j. Now assume that this equation is also satisfied for a greater value of j. Since the last argument is valid for any non-negative integer i, the total number of roots of $q_n^{(\boldsymbol{\lambda})}(x)$ would

be greater than $\sum_{k \geqslant 1} \lambda'_k = n$ that is impossible, since $q_n^{(\boldsymbol{\lambda})}(x)$ is a polynomial of degree at most n. Thus, $x = i$ is a root of $q_n^{(\boldsymbol{\lambda})}(x)$ with multiplicity λ'_{i+1} and consequently

$$q_n^{(\boldsymbol{\lambda})}(x) = c_n^{(\boldsymbol{\lambda})} \prod_{i=0}^{n-1} (x - i)^{\lambda'_{i+1}}, \tag{6.51}$$

where $c_n^{(\boldsymbol{\lambda})}$ is a constant.

To determine this constant coefficient, one notes that for large values of x, Equation (6.29) gives rise to

$$(-1)^n p_1^n = \sum_{\boldsymbol{\lambda} \vdash n} c_n^{(\boldsymbol{\lambda})} m_{\boldsymbol{\lambda}}. \tag{6.52}$$

By multinomial theorem, the left hand side of the equation above is

$$\sum_{\substack{k_1, \ldots, k_{mN} \geqslant 0 \\ k_1 + \cdots + k_{mN} = n}} (-1)^n \frac{n!}{k_1! \cdots k_{mN}!} \, w_1^{k_1} \cdots w_{mN}^{k_{mN}} = \sum_{\boldsymbol{\lambda} \vdash n} (-1)^n \binom{n}{\lambda_1, \ldots, \lambda_n} m_{\boldsymbol{\lambda}}, \tag{6.53}$$

if $mN \geqslant n$. Hence

$$c_n^{(\boldsymbol{\lambda})} = (-1)^n \binom{n}{\lambda_1, \ldots, \lambda_n}, \tag{6.54}$$

and the theorem is proved. $\qquad \square$

6.2.5   Behavior of $r_n^{(x)}$ under Translations

This section is devoted to investigate the behavior of polynomials $r_n^{(x)}$ under the translation operator

$$L^- = \sum_{i=1}^{mN} \partial_i, \tag{6.55}$$

where ∂_i refers to the derivative with respect to the ith variable. It is straight forward to see that, for any positive integer r, L^- acts on the rth power-sum polynomial p_r according to

$$L^- p_r = r \, p_{r-1}. \tag{6.56}$$

Recall that p_0 is the number of variables mN. Using the Leibniz rule for differentiating a product of two functions a certain number of times, one can show that for any two functions f and g,

$$L^-(fg) = (L^- f)g + f(L^- g). \tag{6.57}$$

The next theorem shows how L^- acts on $r_n^{(x)}$. To prove this theorem one needs a recursion relation for $r_n^{(x)}$ that is the content of Lemma 6.10.

Lemma 6.10. *The polynomials $r_n^{(x)}$ satisfy the recursion relation*

$$r_n^{(x)} = -x(n-1)! \sum_{i=1}^{n} p_i \frac{r_{n-i}^{(x)}}{(n-i)!}, \tag{6.58}$$

where p_i is the ith power-sum symmetric polynomial.

Proof. From Equations (4.115) and (6.11) one can get the following equation

$$r_n^{(1)} = -(n-1)! \sum_{i=0}^{n-1} r_i^{(1)} \frac{p_{n-i}}{i!}. \tag{6.59}$$

Applying $\mathcal{C}_m$ on both sides of Equation (6.59) and using Equations (6.1) and (6.8) and then relabelling m by x and reordering the resultant sum yields Equation (6.58). $\qquad\square$

Theorem 6.11. *The operator L^- acts on $r_n^{(x)}$ as follows*

$$L^- r_n^{(x)} = n(n-1-xp_0) \, r_{n-1}^{(x)}, \tag{6.60}$$

where p_0 is the 0th power-sum.

Proof. By Equation (6.57), if $p_{\boldsymbol{\lambda}} = \prod_{i \geqslant 1} p_{\lambda_i}$,

$$L^- p_{\boldsymbol{\lambda}} = \sum_{j \geqslant 1} m_j(\boldsymbol{\lambda}) j \, p_{j-1} p_{\boldsymbol{\lambda}-\{j\}}, \tag{6.61}$$

where $m_j(\boldsymbol{\lambda})$ is the multiplicity of j in $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda} - \{j\}$ denotes the partition derived from $\boldsymbol{\lambda}$ by deleting one part that is equal to j. Acting by L^- on both sides of Equation (6.2) and employing Equation (6.61) gives

$$L^- r_n^{(x)} = n! \sum_{\boldsymbol{\lambda} \vdash n} \left(\frac{(-x)^{l(\boldsymbol{\lambda})}}{z_{\boldsymbol{\lambda}}} \sum_{j=1}^{n} m_j(\boldsymbol{\lambda}) j \, p_{j-1} p_{\boldsymbol{\lambda}-\{j\}} \right). \tag{6.62}$$

Now let us change the summation index from $\boldsymbol{\lambda}$ to $\boldsymbol{\mu} = \boldsymbol{\lambda} - \{j\}$. Then

$$l(\boldsymbol{\lambda}) = l(\boldsymbol{\mu}) + 1, \tag{6.63}$$
$$m_j(\boldsymbol{\lambda}) = m_j(\boldsymbol{\mu}) + 1, \tag{6.64}$$
$$z_{\boldsymbol{\lambda}} = z_{\boldsymbol{\mu}}(m_j(\boldsymbol{\mu}) + 1)j, \tag{6.65}$$

and Equation (6.62) can be written as

$$L^- r_n^{(x)} = (-x)n! \sum_{j=1}^{n} \left(p_{j-1} \sum_{\boldsymbol{\mu} \vdash (n-j)} \frac{(-x)^{l(\boldsymbol{\mu})}}{z_{\boldsymbol{\mu}}} p_{\boldsymbol{\mu}} \right). \tag{6.66}$$

The sum above can be split into two parts, one part corresponding to $j = 1$ and the other part corresponding to the rest of j values. By Equation (6.2), the first part is

$$(-x)n!\, p_0 \sum_{\boldsymbol{\mu} \vdash (n-1)} \frac{(-x)^{l(\boldsymbol{\mu})}}{z_{\boldsymbol{\mu}}} p_{\boldsymbol{\mu}} = (-x)np_0\, r^{(x)}_{n-1}, \qquad (6.67)$$

and, by Equation (6.58), the second part is

$$(-x)n! \sum_{j=2}^{n} \left(p_{j-1} \sum_{\boldsymbol{\mu} \vdash (n-j)} \frac{(-x)^{l(\boldsymbol{\mu})}}{z_{\boldsymbol{\mu}}} p_{\boldsymbol{\mu}} \right) = (-x)n! \sum_{j=2}^{n} \left(p_{j-1} \frac{r^{(x)}_{n-j}}{(n-j)!} \right)$$

$$= n(n-1)\, r^{(x)}_{n-1}. \qquad (6.68)$$

Adding both sides of Equations (6.66) and (6.68) proves the theorem. $\square$

Epilogue

The main goal of our work was to provide evidence for Conjecture 1.1 by considering a FQH system consisting of N electrons in its ground state modeled by a generic Laughlin state Ψ_m and show that the rank of the reduced density operator, obtained by dividing the system into two parts by means of the particle cut scheme, is equal to the number of quasi-hole states, which in turn forms an upper bound on this rank. The hard part of this problem is showing that this upper bound is satisfied.

Our strategy in this regard was to find a weak Schmidt decomposition of Ψ_m and then use Theorem 3.3 to determine the demanded rank. The technique we used was to define a particular transformation (5.6), referred as clustering transformation in the text. Roughly speaking, this mathematical trick enabled us to look for the weak Schmidt decomposition of a copy of Ψ_1 instead of Ψ_m but for a larger number of variables. This seemed easier to do, since then we were able to use a known decomposition rule in the theory of symmetric polynomials expressed as the second part of Theorem 4.4. Finally we came to Equation (5.15) as a candidate for the weak Schmidt decomposition of Ψ_m. According to Theorem 3.3, for Equation (5.15) to be a weak Schmidt decomposition one should prove that the set (5.16) is linearly independent. This motivated us to look at symmetric polynomials in mN variables that vanish under clustering transformation $\mathcal{C}_m$. Probably one of the alternatives in this regard was to try to characterize these polynomials with respect to their total degree. We succeeded to do that by constructing a new set of generators for symmetric polynomials in mN variables that helped us to prove that the *total* degree of any non-zero symmetric polynomial in mN variables that vanishes under $\mathcal{C}_m$ is at least $N + 1$.

Although this is a very interesting result on its own right but, unfortunately, it turned out that it is not sufficient for our initial goal. To be able to prove the Conjecture 1.1, one needs to show that the *degree* in any variable of any non-zero symmetric polynomial in mN variables that vanishes under $\mathcal{C}_m$ is at least $N + 1$. This puts the physical Conjecture 1.1 for the Laughlin case equivalent to the mathematically formulated Conjecture 1.2.

Naturally afterwards, we tried to discover some useful properties of the new introduced generators to be able to go further and complete the proof.

Although we did not succeed, but this attempt gave rise to some interesting results for the properties of these generators that might be useful for more investigations on the problem. Probably the most interesting result is that in the set of these generators, the ones that vanish under $\mathcal{C}_m$ have degree at least $N + 1$, which is a special case of the more general Conjecture 1.2.

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Accompanied Paper

On the particle entanglement spectrum of the Laughlin states

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Abstract

The study of the entanglement entropy and entanglement spectrum has proven to be very fruitful in identifying topological phases of matter. Typically, one performs numerical studies of finite-size systems. However, there are few rigorous results in this regard. We revisit the problem of determining the rank of the 'particle entanglement spectrum' (PES) of the Laughlin states. We reformulate the problem into a problem concerning the ideal of symmetric polynomials that vanish under the formation of several clusters of particles. We introduce an explicit generating set of this ideal, and we prove that polynomials in this ideal have a total degree that is bounded from below. We discuss the difficulty in proving the same bound on the degree of any of the variables, which is necessary to determine the rank of the PES.

Keywords: entanglement spectrum, quantum Hall effect, Laughlin state

1. Introduction

The study of topological phases of matter has benefited greatly from considering the entanglement properties of the ground states of topological phases. The work of Kitaev and Preskill [1] and of Levin and Wen [2] revealed that the entanglement entropy is a good probe of the topological nature of a system and provides a measure for the particle content of the topological phase [3]. The entanglement entropy of a pure quantum state $|\Psi\rangle$ relative to a bipartite partition of the total Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ provides a measure of the entanglement of $|\Psi\rangle$. The entanglement entropy is defined as the Von Neumann entropy S of the reduced density matrix of either one of the two parts. For instance

$$S = -\text{Tr}\left(\rho_A \log \rho_A\right), \tag{1}$$

where $\rho_A = \text{Tr}_B(|\Psi\rangle\langle\Psi|)$.

In the context of the fractional quantum Hall (FQH) effect, various ways to partition the Hilbert space were proposed [4]. Of particular importance, is the spatial partitioning scheme in which the system is split into two regions A and B separated by a real-space cut of length L. For a system exhibiting topological order the real-space entanglement entropy is of the form [2, 3]

$$S = \alpha L - \gamma + \cdots, \tag{2}$$

where $\cdots$ stands for subdominant terms as L becomes large. The subdominant term γ is universal, and depends only on the nature of the topological phase. It bears the name *topological entanglement entropy*, and is a measure for the particle content of the topological phase [3]. The first term αL, while non-universal, means that the amount of entanglement is proportional to the length of the boundary separating the two regions. This property called *area law* has appeared in various areas of physics, such as black-hole physics and quantum information. For a quantum many-body state, this property is of particular importance since it opens the way to extremely efficient numerical simulations such as the density matrix renomalization group [5] and matrix product states [6] methods. For FQH state this avenue of research was successfully undertaken [7–9] and opened the way to a reliable microscopic calculation of quasi-holes properties such as radius and braiding [10].

Although the real-space cut is of paramount importance in the study of topological phases of matter, there are other natural ways to partition a quantum Hall system: the *orbital cut*, and the *particle cut* [4]. While, in principle, the entanglement entropy behaves according to the area law equation (2) only for real-space cuts, it was numerically observed [11] that the area law is also valid for orbital cuts. In this paper we will concentrate on the particle cut, in which one numbers the (identical) particles constituting the phase (for instance, the electrons in the quantum Hall case), and one declares the particles numbered 1, 2, …, N_A to belong to subsystem A, while the remaining particles numbered $N_A + 1$, $N_A + 2$, …, N belong to subsystem B. The spectrum of the reduced density matrix obtained by tracing out the particles in subsystem B is the 'particle entanglement spectrum' (PES) [12].

While the entanglement entropy S provides a good probe of topological order, the topological entanglement entropy γ does not determine unambiguously the universality class of the topological state. Li and Haldane [13] realized that the spectrum of $H_A = -\log \rho_A$, which is called the entanglement Hamiltonian, itself contains much more information than the entanglement entropy. They proposed to use the low lying part of this *entanglement spectrum* as a 'fingerprint' of the topological phase. To be more specific, under a bipartition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, a pure quantum state $|\Psi\rangle$ admits a Schmidt decomposition

$$|\Psi\rangle = \sum_i e^{-\xi_i/2} \left|\psi_i^A\right\rangle \otimes \left|\psi_i^B\right\rangle, \tag{3}$$

where the $e^{-\xi_i/2}$'s are positive numbers called the Schmidt singular values, while $|\psi_i^A\rangle$ and $|\psi_i^B\rangle$ form orthonormal sets in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The reduced density matrix is then simply

$$\rho_A = \sum_i e^{-\xi_i} \left|\psi_i^A\right\rangle \left\langle\psi_i^A\right| = e^{-H_A}. \tag{4}$$

The entanglement spectrum is the set of all *entanglement energies* ξ_i. The bipartition can be chosen to preserve as much symmetry as possible, which in turn yields quantum numbers for the ξ_i's, such as the momentum along the cut. Li and Haldane observed that—per momentum

sector—the number of entanglement energies reproduces exactly the number of gapless edge modes. They proposed that tracing out the degrees of freedom of part *B* introduces a virtual edge for part *A*. The Li-Haldane conjecture is therefore two-fold. For a FQH state in the thermodynamic limit:

 (i) the entanglement energies and edge modes have the same counting,
 (ii) the entanglement spectrum is proportional to the (edge) CFT spectrum.

It is now understood that (ii) can only hold in the case of a real-space cut, which maintains locality along the cut [14–16]. For an orbital cut the entanglement Hamiltonian H_A has no reason to be local. On the other hand the point (i) holds irrespective of the cut for model wave functions that can be written as correlation functions in a CFT. Such wave functions are precisely of MPS form [17], and the CFT Hilbert space provides a one-to-one mapping [18] between edge modes and entanglement energies.

While the agreement between the counting of the number of modes in the entanglement spectrum and the counting of the edge excitations is well understood, in practice, this fingerprint is used for finite-size systems. The entanglement counting develops finite-size effects, which naively have no structure. However, it has been conjectured and numerically substantiated [19] that there is a counting principle underlying the finite-size entanglement counting of model states. Before we turn to the main topic of this paper, namely the PES for the Laughlin states, we mention that the entanglement entropy and spectra are actively studied, see [20–26] for some studies in the context of the quantum Hall effect.

We now consider the ground state $|\Psi\rangle$ of a model quantum Hall state, such as the Laughlin [27] or Moore–Read [28] state, that are the exact zero energy states of a model Hamiltonian. For a given number N of particles, this is the unique zero-energy state of a model Hamiltonian that occurs at the following number of flux quanta

$$N_\phi = \frac{1}{\nu}N - S, \tag{5}$$

where ν is the filling fraction and S is the shift. Now suppose that the N particles are divided into two groups, group *A* containing N_A of the particles, and group *B* containing the rest $N_B = N - N_A$ of them. Without any loss of generality, we can assume that $N_A \leqslant N_B$. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_{N_A})$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_{N_B})$ be the coordinates of particles in *A* and *B*, respectively. The Schmidt decomposition of the wave function $\Psi(\boldsymbol{z})$, that is

$$\Psi(\boldsymbol{z}) = \sum_i e^{-\xi_i/2} \psi_i^A(\boldsymbol{x}) \psi_i^B(\boldsymbol{y}), \tag{6}$$

involves wave functions $\psi_i^A(\boldsymbol{x})$ for N_A particles. After tracing out the particles of part *B*, we are left with a reduced system of N_A particles, but the amount of flux remains the same, namely

$$N_\phi = \frac{1}{\nu}N_A - S + \Delta N_\phi, \tag{7}$$

where $\Delta N_\phi = \nu^{-1} N_B$. The presence of this excess flux ΔN_ϕ indicates that we should view the reduced system as one with N_A particles, in the presence of quasi-hole excitations. For a *real* system with N_A particles, and ΔN_ϕ excess flux quanta the number of zero-energy states of the model Hamiltonian (which we will call the number of quasi-hole states) can often be obtained exactly [29–33]. For instance, in the case of the $\nu = 1/m$ Laughlin case, quasi-hole states of N_A particles in ΔN_ϕ excess flux are of the form

$$\psi_i(\boldsymbol{x}) = P_i(\boldsymbol{x}) \prod_{i<k} (x_i - x_k)^m, \tag{8}$$

where $P_i(\boldsymbol{x})$ is a symmetric polynomial in N_A variables with degree in each variable at most ΔN_ϕ. The number of quasi-hole states is therefore

$$\begin{pmatrix} \Delta N_\phi + N_A \\ N_A \end{pmatrix}. \tag{9}$$

This number forms an upper bound for the rank of the reduced density matrix [4, 11, 23].

From numerical investigations, it is known that in all cases considered, this upper bound is in fact reached[5]. This observation has led to the 'rank saturation' conjecture, which can be thought of as a finite-size version of the Li-Haldane conjecture, namely, the entanglement level counting of the PES of a model state is equal to the number of bulk quasi-hole states. This means that the states $\psi_i^A(\boldsymbol{x})$ appearing in the Schmidt decomposition of $\Psi(\boldsymbol{z})$ span all the quasi-hole states of N_A particles in ΔN_ϕ excess flux. Proving analytically that this upper bound is indeed reached has proven to be a difficult problem.

In this paper, we revisit this problem for the general $\nu = \frac{1}{m}$ Laughlin states. We start by considering the $\nu = 1$ Laughlin state, which is simply the Slater determinant of the completely filled lowest Landau level. We explain how to obtain the rank of the reduced density matrix of the PES in this case. To do so, we will make some use of the properties of symmetric polynomials. To get a grip on the $\nu = \frac{1}{m}$ Laughlin states, we then use the following strategy. After partitioning the particles into two sets A and B, we 'split' the N_B particles in part B into mN_B particles, and consider the $\nu = 1$ Laughlin state of the system thus obtained. For this system, we already obtained the rank of the reduced density matrix. If one can show that clustering the mN_B particles into groups of size m, does not lead to a smaller rank of the reduced density matrix, one deduces the rank of the reduced density matrix for the $\nu = \frac{1}{m}$ Laughlin state, and shows that the upper bound is indeed reached.

The hard step in the strategy outlined above is to show that the clustering of the mN_B particles into N_B groups of size m does not reduce the rank of the reduced density matrix. Proving this statement turns out to be highly non-trivial. As we explain in the main text, one has to show that there is no (non-zero) symmetric polynomial in mN_B variables that vanishes under the formation of N_B groups of variables each of size m, and whose degree in any of the variables is N_B or less. Although we did not fully succeed in proving this statement, we did make substantial progress. In particular, we constructed an explicit generating set of the ideal of polynomials that vanish under this clustering. Using this construction we were able to show that a non-zero symmetric polynomial in mN_B variables that vanishes under the formation of N_B groups of variables each of size m must have a *total* degree at least $N_B + 1$. Proving this weaker statement is already a non-trivial result, mainly because the positions of the various clusters can be arbitrary, which means that the clustering condition is non-local. Moreover, we were able to prove that all polynomials in the generating set have a degree at least $N_B + 1$.

The outline of the article is as follows. In section 2, we introduce the notion of partitions, and several types of symmetric polynomials, that we make use of throughout the article. The PES of the $\nu = 1$ Laughlin state is discussed in section 3. We continue in section 4 by explaining how the result for $\nu = 1$ can be used to make progress for the $\nu = 1/m$ Laughlin states, and recast the problem in terms of clustering properties of symmetric polynomials. In section 5, we prove that the *total* degree of polynomials that vanish under clustering is

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[5] Provided no additional trivial constraint, coming from the Hilbert space of the reduced system, is present. The particles of the simplest systems for which this can happen have spin $S = 1$.

bounded from below, and provide an explicit construction for such polynomials in general. In section 7, we make some comments on why it is much harder to prove that not only the total degree, but also the degree of any variable for polynomials that vanish under the clustering is bounded from below. In addition, we provide a proof for the statement in the case where one forms two clusters of size m. Finally, we discuss our results in section 8. In the appendix A, we derive some properties of the polynomials which are used in section 5, and in appendix B we provide an alternate set of polynomials that can be used in the proof of section 5.

2. Some notations

In this section we introduce some definitions and notations that are used in what follows. We start by introducing the notion of partitions, which play a central role in the theory of symmetric polynomials. For a general introduction to the subject of partitions, we refer to [35] and for the theory of symmetric polynomials to [34].

2.1. Partitions

For a positive integer $\mathcal{D}$, a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ of strictly positive integers $\lambda_1, \lambda_2, ..., \lambda_r$ is called an *r-partition* of $\mathcal{D}$ if $\sum_{i=1}^{r} \lambda_i = \mathcal{D}$. The λ_i's are the *parts* of λ, and r is called the *length* of λ, which is denoted by $l(\lambda)$. We call $\mathcal{D}$ the *weight* of λ, which is denoted by $|\lambda|$. We write $\lambda \vdash \mathcal{D}$ to indicate that λ is a partition of $\mathcal{D}$. By convention, $\lambda = \varnothing$ is the only partition of zero which we call the *empty* partition. The number of parts of partition λ that are equal to a given integer j is denoted by $n_j(\lambda)$ or simply n_j. We also define

$$z_\lambda = \prod_{j=1}^{\lambda_1} j^{n_j} n_j!. \tag{10}$$

Finally, the set of all partitions of $\mathcal{D}$ is denoted by $\mathrm{Par}(\mathcal{D})$. It is not too hard to convince oneself (see [35]) that the number of partitions with at most r parts and each part at most d is equal to $\binom{r+d}{r}$.

2.2. Symmetric polynomials

In what comes, we will be dealing with the ring Λ_N of *symmetric polynomials* in N variables. A polynomial P is called a symmetric polynomial in N variables, if for all permutations σ of $\{1, ..., N\}$,

$$P\left(x_{\sigma(1)}, ..., x_{\sigma(N)}\right) = P(x_1, ..., x_N). \tag{11}$$

The degree d of a symmetric polynomial is simply the degree in one of its variables.

A polynomial $P(x_1, ..., x_N)$ is called homogeneous of total degree $\mathcal{D}$, if for any real number l,

$$P\left(l\, x_1, ..., l\, x_N\right) = l^{\mathcal{D}} P(x_1, ..., x_N). \tag{12}$$

For instance, the polynomial $P(x_1, x_2) = x_1^2 x_2 + x_1 x_2^2$ is a homogeneous symmetric polynomial of degree $d = 2$ and total degree $\mathcal{D} = 3$.

There are different bases that one can consider for Λ_N. A natural one, is given by the set of, so-called, *symmetric monomials*. Given a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ with $r \leqslant N$, the symmetric monomial $m_\lambda(x_1, ..., x_N)$ is defined as

$$m_\lambda(x_1, \ldots, x_N) := \sum_\sigma x_{\sigma(1)}^{\lambda_1} x_{\sigma(2)}^{\lambda_2} \cdots x_{\sigma(r)}^{\lambda_r} x_{\sigma(r+1)}^0 \cdots x_{\sigma(N)}^0, \tag{13}$$

where the sum is over all *distinct* permutations σ of the parts of λ, and it is defined to be 1 if λ is the empty partition. On the other hand if $r > N$ we set $m_\lambda(x_1, \ldots, x_N) = 0$. For example,

$$m_{(2,1,1)}(x_1, x_2, x_3) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2,$$

$$m_{(2,1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2,$$

$$m_{(2,1,1)}(x_1, x_2) = 0. \tag{14}$$

When studying rank saturation of the PES for the Laughlin state, finite-size effects imply an upper bound for the degree of polynomials. We will therefore be led to consider the space Λ_N^d of symmetric polynomials in N variables, with degree (in each of the variables) at most d. A basis for this space is given by the symmetric monomials $m_\lambda(x_1, \ldots, x_N)$ corresponding to partitions λ with at most N parts and each part at most d. Therefore, we have

$$\dim\left(\Lambda_N^d\right) = \begin{pmatrix} N + d \\ N \end{pmatrix}. \tag{15}$$

Another important family of symmetric polynomials is the set of *elementary symmetric polynomials*. The elementary symmetric polynomials that are labelled by an integer n are defined in terms of symmetric monomials as $e_n := m_{\underbrace{(1,\ldots,1)}_{n \text{ ones}}}$. For instance,

$$e_0(x_1, x_2, x_3) = 1,$$

$$e_1(x_1, x_2, x_3) = x_1 + x_2 + x_3,$$

$$e_2(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3,$$

$$e_3(x_1, x_2, x_3) = x_1 x_2 x_3,$$

$$e_{n \geqslant 4}(x_1, x_2, x_3) = 0. \tag{16}$$

For a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, the elementary symmetric polynomial e_λ is defined as $e_\lambda := e_{\lambda_1} \cdots e_{\lambda_r}$. As an example,

$$e_{(2,1,1)}(x_1, x_2) = e_2(x_1, x_2) e_1(x_1, x_2) e_1(x_1, x_2)$$

$$= x_1 x_2 (x_1 + x_2)^2. \tag{17}$$

It is known that the set of all polynomials $e_\lambda(x_1, \ldots, x_N)$, where λ is a partition with at most d parts and each part at most N, forms a basis of the space Λ_N^d.

Lastly, the *power sum symmetric polynomials*, defined as

$$p_i(x_1, \ldots, x_N) := x_1^i + \cdots + x_N^i, \tag{18}$$

are of special importance. In fact, the set $\{p_1, p_2, \ldots, p_N\}$ generates Λ_N. This means that any symmetric polynomial P in N variables can be written as a polynomial in $(p_1, \ldots, p_N)$. In other words, the set of polynomials $p_\lambda := p_{\lambda_1} \cdots p_{\lambda_r}$, where λ is a partition with each part at most N, forms a basis of Λ_N. For example,

$$e_2 = \sum_{i<j} x_i x_j = \frac{p_1^2 - p_2}{2}, \tag{19}$$

independent of the number of variables N. Most importantly, the decomposition of any symmetric polynomial P in N variables as a polynomial in $(p_1, \ldots, p_N)$ is unique. One should note that, unlike for the symmetric monomials m_λ and the elementary symmetric polynomials e_λ, there is no natural restriction on λ such that the corresponding p_λ's form a basis for Λ_N^d.

3. The $\nu = 1$ state

To obtain the rank of the reduced density matrix of the Laughlin states in the case of the 'particle cut', we start by considering the simplest case, the $\nu = 1$ Laughlin state, which is just a single Slater determinant,

$$\Psi_{\nu=1}(z_1, \ldots, z_N) = \prod_{1 \leqslant i < j \leqslant N} (z_i - z_j), \tag{20}$$

up to a geometry-dependent Gaussian factor. For instance, the plane and sphere geometry give rise to different Gaussian factors, inherited from the respective inner products. However, for our purposes the precise form of the Gaussian factor is irrelevant. The results presented in this paper involves only the notion of linear independence, and does not refer to the notion of orthogonality. As a consequence, the underlying inner product plays no role and our result is equally valid on the plane, sphere, and cylinder.

Now suppose that the N particles are divided into two groups A, containing N_A of the particles, and B containing $N_B = N - N_A$ particles. At this stage we do not assume $N_A \leqslant N_B$. Let us rename the coordinates of particles in A and B to $\boldsymbol{x} = (x_1, x_2, \ldots, x_{N_A})$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_{N_B})$, respectively. The rank of the reduced density matrix in the case of such a particle cut can be obtained from a decomposition of the wave function $\Psi(z)$ of the form

$$\Psi(z) = \sum_i \psi_i^A(\boldsymbol{x}) \psi_i^B(\boldsymbol{y}), \tag{21}$$

where the set of wave functions ψ_i^A (resp. ψ_i^B) are independent. Note that this is not quite a Schmidt decomposition since we do not demand the ψ_i^A's to form an orthonormal set. Although this is not a Schmidt decomposition, the number of terms in the sum is equal to the Schmidt rank, or equivalently, to the rank of the reduced density matrix. Therefore, we will call the decomposition (21) a Schmidt decomposition, although strictly speaking this is an abusive notation.

Before we explicitly write the $\nu = 1$ Laughlin state in such a 'Schmidt-decomposed' form, we note that we can obtain the rank of the reduced density matrix in the $\nu = 1$ case in a straightforward way. This state is simply obtained by filling the Landau orbitals from 0 up to $N_\Phi = N - 1$

$$\left| \Psi_{\nu=1} \right\rangle = \left| 111 \cdots 111 \right\rangle. \tag{22}$$

The Schmidt decomposition relative to the particle cut scheme described above amounts to choose N_A out of the N particles

$$\left| \Psi_{\nu=1} \right\rangle \propto \left| \underbrace{111 \cdots 11}_{N_A} 00 \cdots 0 \right\rangle \otimes \left| 000 \cdots 0011 \cdots 1 \right\rangle$$

$$+ \left| \underbrace{111 \cdots 10}_{N_A} 10 \cdots 0 \right\rangle \otimes \left| 000 \cdots 0101 \cdots 1 \right\rangle + \cdots$$

$$+ \left| \underbrace{000 \cdots 00}_{N_A} 11 \cdots 1 \right\rangle \otimes \left| 111 \cdots 1100 \cdots 0 \right\rangle, \tag{23}$$

which means that the rank of the reduced density matrix is given by the number of ways in which the N_A particles of system A can be divided over the number of orbitals. The number of orbitals is given by $N = N_A + N_B$, so we obtain that the rank of the reduced density matrix is

given by $\begin{pmatrix} N_A + N_B \\ N_A \end{pmatrix}$. We note that the same result can be obtained directly from the wave function [15], which is a single Slater determinant $\Psi_{\nu=1}(\boldsymbol{z}) = \prod_{i<j} (z_i - z_j)$.

It is instructive to perform a more explicit Schmidt decomposition of the $\nu = 1$ Laughlin wave function. We start by writing the state explicitly in terms of the variables x_i and y_i of groups A and B, respectively. Dropping the exponential factors, we have

$$\Psi_{\nu=1}(\boldsymbol{z}) = \Psi_{\nu=1}(\boldsymbol{x}) \left(\prod_{i,j} \left(x_i - y_j \right) \right) \Psi_{\nu=1}(\boldsymbol{y}). \tag{24}$$

We are going to use the following result [34]

$$\prod_{i=1}^{N_A} \prod_{j=1}^{N_B} \left(1 + x_i y_j \right) = \sum_\lambda m_\lambda(\boldsymbol{x}) e_\lambda(\boldsymbol{y}), \tag{25}$$

where the sum is over all partitions λ with maximally N_A parts, and each part being maximally N_B, i.e., all partitions which fit in a rectangle of height N_A and width N_B. Thus

$$\prod_{i,j} \left(x_i - y_j \right) = \sum_\lambda (-1)^{|\lambda|} m_{\bar{\lambda}}(\boldsymbol{x}) e_\lambda(\boldsymbol{y}). \tag{26}$$

Here, we used the relation

$$\left(\prod_i x_i^{N_B} \right) m_\lambda(-1/\boldsymbol{x}) = (-1)^{|\lambda|} m_{\bar{\lambda}}(\boldsymbol{x}), \tag{27}$$

where the partition $\bar{\lambda}$ stands for the complement of λ with respect to the rectangle of height N_A and width N_B. In addition, $(-1/\boldsymbol{x})$ is shorthand for $(-1/x_1, \ldots, 1/x_{N_A})$. As an example, it is shown in figure 1 that for $N_A = 3$, $N_B = 4$ and $\lambda = (2, 1)$, one finds $\bar{\lambda} = (4, 3, 2)$. We then obtain a Schmidt decomposition for the $\nu = 1$ Laughlin state

$$\Psi_{\nu=1}(\boldsymbol{z}) = \sum_\lambda (-1)^{|\lambda|} q_\lambda^A(\boldsymbol{x}) q_\lambda^B(\boldsymbol{y}), \tag{28}$$

where

$$q_\lambda^A(\boldsymbol{x}) = m_{\bar{\lambda}}(\boldsymbol{x}) \Psi_{\nu=1}(\boldsymbol{x}), \tag{29}$$

and

$$q_\lambda^B(\boldsymbol{y}) = e_\lambda(\boldsymbol{y}) \Psi_{\nu=1}(\boldsymbol{y}). \tag{30}$$

A few remarks on this formula are in order here. The number of terms in the sum is most important here. The sum over λ, is over all partitions with maximally N_A parts, and each part being maximally N_B, i.e., all partitions which fit in a rectangle of height N_A and width N_B. There are $\begin{pmatrix} N_A + N_B \\ N_A \end{pmatrix}$ such partitions. The rank of the reduced density matrix of the $\nu = 1$ Laughlin state is thus given by

$$\begin{pmatrix} N_A + N_B \\ N_A \end{pmatrix}, \tag{31}$$

and we recover the dimension of $\Lambda_{N_A}^{N_B}$. It is straightforward to check that this is also the dimension of the set of anti-symmetric polynomials in N_A variables with maximum degree
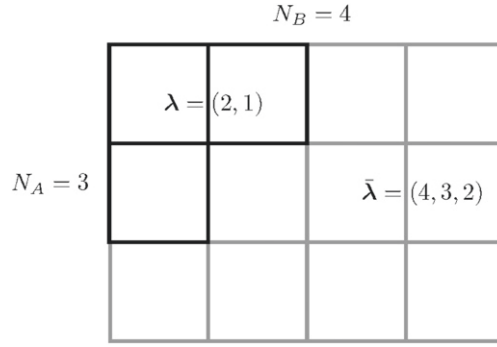
Figure 1. The relation between the partition λ and its complement $\bar{\lambda}$ for given N_A and N_B.

$N = N_A + N_B$, which is nothing but the space of 'quasi-hole' states for the non-interacting $\nu = 1$ case. The set of polynomials q_λ^A (resp. q_λ^B) forms a basis for the space of anti-symmetric polynomials in N_A (resp. N_B) variables with maximum degree N. Note that this result is symmetric under exchange of A and B, and in particular it holds whether or not $N_A \leqslant N_B$. This is a particularity of the $\nu = 1$ case and it will no longer be true for the $\nu = \frac{1}{m}$ Laughlin state with $m > 1$.

4. Schmidt decomposition of the $\nu = \frac{1}{m}$ Laughlin state

We are now going to compute the rank of the reduced density matrix for the generic $\nu = 1/m$ Laughlin state

$$\Psi_m(z_1, \ldots, z_N) = \prod_{1 \leqslant i < j \leqslant N} (z_i - z_j)^m. \tag{32}$$

As usual we divide the particles into two groups A and B, containing N_A and $N_B = N - N_A$ of them, respectively, and we assume that $N_A \leqslant N_B$. We are interested in obtaining a Schmidt decomposition of this state. As for the $\nu = 1$ case, we can write

$$\Psi_m(z) = \Psi_m(x) \left(\prod_{i=1}^{N_A} \prod_{j=1}^{N_B} \left(x_i - y_j \right)^m \right) \Psi_m(y). \tag{33}$$

Proving that the rank of the PES for the $\nu = 1/m$ Laughlin state is saturated boils down to finding a Schmidt decomposition for the wave function (33) and counting the number of terms in the decomposition. As in the $\nu = 1$ case, one needs to take care of only the middle term of the wave function (33)

$$\Phi_m(x, y) := \prod_{i=1}^{N_A} \prod_{j=1}^{N_B} \left(x_i - y_j \right)^m. \tag{34}$$

To do so we start from

$$\Phi_1(x, w) = \prod_{i=1}^{N_A} \prod_{j=1}^{mN_B} (x_i - w_j), \tag{35}$$

for $\boldsymbol{w} = (w_1, \ldots, w_{mN_B})$. From (26),

$$\Phi_1(\boldsymbol{x}, \boldsymbol{w}) = \sum_\lambda (-1)^{|\lambda|} m_{\bar\lambda}(\boldsymbol{x}) e_\lambda(\boldsymbol{w}), \tag{36}$$

but this time the sum is over all partitions λ which fit in a rectangle of height N_A and width mN_B. We then relate Φ_m to Φ_1 through the clustering transformation, which is a linear transformation from Λ_{mN_B} to Λ_{N_B} defined as follow.

To a symmetric polynomial $P(\boldsymbol{w})$ in mN_B variables, we associate the polynomial $C_m(P(\boldsymbol{y}))$ in N_B variables obtained by regrouping the particles into clusters of m, i.e.

$$(C_m P)(y_1, y_2, \ldots, y_{N_B}) = P\left(\underbrace{y_1, y_1, \ldots, y_1}_{m}, \underbrace{y_2, y_2, \ldots, y_2}_{m}, \ldots, \underbrace{y_{N_B}, y_{N_B}, \ldots, y_{N_B}}_{m} \right). \tag{37}$$

It is easy to see that after clustering Φ_1 becomes Φ_m, i.e.

$$C_m(\Phi_1) = \Phi_m. \tag{38}$$

Applying the clustering transformation to both sides of equation (36) results in

$$\Phi_m(\boldsymbol{x}, \boldsymbol{y}) = \sum_\lambda (-1)^{|\lambda|} m_\lambda(\boldsymbol{x}) C_m(e_\lambda)(\boldsymbol{y}). \tag{39}$$

As mentioned earlier, the sum is over all partitions λ with maximally N_A parts, and each part being maximally mN_B. There are $\begin{pmatrix} N_A + mN_B \\ N_A \end{pmatrix}$ such partitions. This is precisely the number of Laughlin quasi-hole states for N_A particles in $\Delta N_\Phi = mN_B$ extra fluxes, and we recover the usual upper bound for the rank of the reduced density matrix.

Rank saturation of the PES for the $\nu = 1/m$ Laughlin state boils down to the following, non-trivial result: the polynomials $C_m(e_\lambda)$ are independent. More precisely, one has to prove that the linear transformation

$$\Lambda_{mN_B}^{N_A} \longrightarrow \Lambda_{N_B}^{mN_A}$$
$$P(\boldsymbol{w}) \longrightarrow C_m(P)(\boldsymbol{y})$$

is injective as long as $N_A \leqslant N_B$. Since $\dim \Lambda_{N_B}^{mN_A} \geqslant \dim \Lambda_{mN_B}^{N_A}$, it is sufficient to show that this linear map has a trivial kernel. Namely, besides $P = 0$, no polynomial in mN_B variables and maximum degree N_A can vanish under the clustering transformation.

5. Clustering properties of symmetric polynomials

In this section we are going to describe the ideal of symmetric polynomials in $q = mN$ variables that vanish under the clustering transformation (37). In particular, we are going to construct a generating set of this subspace, and prove that a non-zero symmetric polynomial in $q = mN$ variables that vanishes under the clustering transformation has a *total* degree $\mathcal{D}$ of at least $N + 1$. We are also going to prove that this symmetric polynomial of minimal total degree is unique (up to a scaling numerical factor).

The statement that the total degree of a symmetric polynomial in $q = mN$ variables that vanishes under the clustering transformation is at least $N + 1$ is a weaker statement than stating that the degree of each variable is at least $N + 1$, but easier to prove. After finishing

the proof of the statement on the total degree, we comment on how one might prove the stronger statement, limiting the degree of the polynomials.

As a warmup, we start with two simple examples, which we will come back to after the proof. We start with the case $m = 2$ and $N = 1$, i.e., we are looking for a symmetric polynomial in two variables y_1 and y_2, of total degree 2, that vanishes when $y_1 = y_2$. It is easy to see that the polynomial has degree two, namely $(y_1 - y_2)^2$.

The case $m = 2$ and $N = 2$ is already more complicated. With some thought, one can construct a total degree 3 symmetric polynomial in four variables $y_1, \ldots, y_4$, that vanishes when $y_1 = y_2$ and $y_3 = y_4$, namely $(y_1 + y_2 - y_3 - y_4)(y_1 - y_2 + y_3 - y_4)$ $(y_1 - y_2 - y_3 + y_4)$. It is already less trivial to convince oneself that no lower degree symmetric polynomial with the same vanishing properties exists. Upon increasing the m and N, even finding polynomials with the correct vanishing properties becomes a hard problem, which is caused by the non-locality of their defining property. Namely, polynomials have to vanish, independent of the position of the various clusters. As we indicated above, we solve this problem in a constructive way.

Our construction is motivated by the following observation. The ring Λ_q of symmetric polynomials in $q = mN$ variables is generated by $\{p_1, \ldots, p_q\}$, and the power sum polynomials p_i have a very simple behaviour under the clustering transformation (37), namely

$$C_m(p_i) = m\, p_i. \tag{40}$$

However, after the clustering transformation there are only N variables left. This means that $\mathcal{A}_N = \{p_1, \ldots, p_N\}$ forms a minimal set of generators, and the polynomials $\{p_{N+1}, \ldots, p_q\}$ are no longer independent after being clustered. The generators p_i are not very convenient to describe the clustering transformation, and this is why we introduce a new set of generators $\tilde{\mathcal{A}}_q = \{\tilde{p}_1, \ldots, \tilde{p}_q\}$ of Λ_q as

$$\tilde{p}_n = \sum_{\boldsymbol{\mu} \vdash n} (-1)^{|\boldsymbol{\mu}|} \left(-\frac{1}{m} \right)^{l(\boldsymbol{\mu})} \frac{p_{\boldsymbol{\mu}}}{z_{\boldsymbol{\mu}}}. \tag{41}$$

Alternatively, the polynomials $\tilde{p}_n$ can be defined in terms of the polynomials $r_n^{(x)}$ of appendix A through $\tilde{p}_n = \dfrac{(-1)^n}{n!} r_n^{(1/m)}$. The main property of these new polynomials is that they behave nicely under clustering:

$$C_m\left(\tilde{p}_n \right) = e_n, \qquad n = 1, \ldots, N, \tag{42}$$

$$C_m\left(\tilde{p}_n \right) = 0, \qquad n > N, \tag{43}$$

as inherited from the properties of $r_n^{(x)}$ described in appendix A. We also introduce the $\tilde{p}_\lambda$ in the usual way as $\tilde{p}_\lambda = \tilde{p}_{\lambda_1} \ldots \tilde{p}_{\lambda_r}$, where r is the length of λ. In terms of these modified power sums $\tilde{p}_n$, it is now relatively easy to describe the ideal of polynomials in Λ_q that vanish under the clustering transformation (37):

Theorem 1. *The set $\{\tilde{p}_\lambda | N + 1 \leqslant \lambda_1 \leqslant q\}$ forms a basis for the ideal of symmetric polynomials in $q = mN$ variables that vanish under the clustering transformation. In other words, this ideal is generated by the set $\{\tilde{p}_{N+1}, \tilde{p}_{N+2}, \ldots, \tilde{p}_q\}$.*

Proof. Suppose that P is a symmetric polynomial in q variables. Because $\tilde{\mathcal{A}}_q$ is a generating set, there exists a polynomial R in q variables such that

$$P = R\left(\tilde{p}_1, \ldots, \tilde{p}_q\right).$$

Generically, there are two kinds of monomials in the polynomial R. Those that depend only on the first N variables $\tilde{p}_1, \ldots, \tilde{p}_N$, and the ones that depend on at least one of the $\tilde{p}_n$, with $n > N$. Accordingly, R can be decomposed uniquely into a sum of two polynomials

$$R\left(\tilde{p}_1, \ldots, \tilde{p}_q\right) = A\left(\tilde{p}_1, \ldots, \tilde{p}_N\right) + B\left(\tilde{p}_1, \ldots, \tilde{p}_q\right).$$

Thus, by construction, $C_m(B) = 0$. It is now straightforward to check that $C_m(R) = 0$ if and only if $A = 0$, since

$$C_m(P) = A(e_1, \ldots, e_N),$$

and the $\{e_1, \ldots, e_N\}$ are algebraically independent in N variables. Therefore the set $\{\tilde{p}_\lambda | N + 1 \leqslant \lambda_1 \leqslant q\}$ is a basis of the kernel of the clustering transformation. ∎

Corollary 1. *The only symmetric polynomial P in mN variables with total degree N or less that vanishes under the clustering transformation is $P = 0$. Moreover, $\tilde{p}_{N+1}$ is the unique (up to an overall factor) symmetric polynomial in q variables and total degree $N + 1$ that vanishes under this clustering.*

Since the modified power sum $\tilde{p}_n$ has total degree n, this corollary follows directly from theorem 1. Let us illustrate this with two simple examples.

Example 1. Consider the simplest non-trivial case where $q = 2$, $N = 1$, and $m = 2$. In this case, the clustering condition is $y_1 = y_2$. The definition of $\tilde{p}_2$ yields

$$\tilde{p}_2 = \frac{1}{8}\left(p_1^2 - 2p_2\right) = -\frac{1}{8}(y_1 - y_2)^2, \tag{44}$$

which reproduces the expected result.

Example 2. As another example, let $q = 4$, $N = 2$, and $m = 2$. This time, clustering conditions are $y_1 = y_2$ and $y_3 = y_4$. We have

$$\tilde{p}_3 = \frac{1}{48}\left(p_1^3 - 6p_2 p_1 + 8p_3\right). \tag{45}$$

For $q = 4$ variables, this is

$$\tilde{p}_3 = \frac{1}{16}\left(y_1 + y_2 - y_3 - y_4\right)\left(y_1 - y_2 + y_3 - y_4\right)\left(y_1 - y_2 - y_3 + y_4\right). \tag{46}$$

We should note that these two examples are not representative for the general case, in the sense that the polynomials $\tilde{p}_{N+1}$ do not generically factorize to a simpler form. For instance, for $N = 3$ and $m = 2$, we have

$$\tilde{p}_4 = \frac{1}{384}\left(p_1^4 - 12p_2 p_1^2 + 12p_2^2 + 32p_1 p_3 - 48p_4\right),$$

which does not have a simple factorized form when restricting to $q = 6$ variables.

Conjecture 1. There is no non-zero symmetric polynomial P in mN variables with degree N or less that vanishes under the clustering transformation.

While we know that the modified power sum $\tilde{p}_n$ has degree n (see appendix A), this is not sufficient to prove this conjecture.

6. *SU*(2) invariance

In the context of the FQH effect, there is a natural action of $SU(2)$ on $\Lambda_N^{N_\Phi}$ coming from the rotational invariance of the sphere. The angular momentum operators on the sphere [36] are

$$L^- = \sum_{i=1}^N \frac{\partial}{\partial z_i}, \tag{47}$$

$$L^3 = \sum_{i=1}^N \left(z_i \frac{\partial}{\partial z_i} - \frac{N_\Phi}{2} \right), \tag{48}$$

$$L^+ = \sum_{i=1}^N \left(z_i N_\Phi - z_i^2 \frac{\partial}{\partial z_i} \right). \tag{49}$$

Every polynomial P in $\Lambda_N^{N_\Phi}$ has a $SU(2)$ symmetric $\Omega(P)$ with opposite angular momentum L^3 given by

$$\Omega(P)(z_1, \ldots, z_N) = \left(\prod_{i=1}^N z_i^{N_\Phi} \right) P\left(1/z_1, \ldots, 1/z_N \right). \tag{50}$$

Under this $\mathbb{Z}_2$ operation, L^- and L^+ are exchanged and $L^3 \to -L^3$.

These linear operators are compatible with the clustering, in the sense that $C_m L^i = L^i C_m$. Note that in these identities the $SU(2)$ operators in the left-hand side (lhs) act in $\Lambda_{mN}^{N_\Phi}$, while in the rhs they act in $\Lambda_N^{mN_\Phi}$:

$$
\begin{array}{ccc}
\Lambda_{mN}^{N_\Phi} & \xrightarrow{\quad L^i \quad} & \Lambda_{mN}^{N_\Phi} \\
\downarrow{\scriptstyle C_m} & & \downarrow{\scriptstyle C_m} \\
\Lambda_N^{mN_\Phi} & \xrightarrow{\quad L^i \quad} & \Lambda_N^{mN_\Phi}
\end{array}
\tag{51}
$$

The same is true for the $\mathbb{Z}_2$ operation Ω. These commuting properties are straightforward to check for L^3, L^-, and for Ω. Therefore, it also holds for $L^+ = \Omega L^- \Omega$. For instance, the clustering transformation clearly preserves the total degree, hence the action of clustering commutes with L^3. Likewise, L^- being the generator of global translations, it commutes with the clustering. The following theorem follows immediately:

Theorem 2. *The ideal of symmetric polynomials in mN variables that vanish under the clustering transformation is invariant under the action of L^i and Ω.*

Corollary 2. *The polynomial $\tilde{p}_{N+1}$ is translationally invariant.*

The polynomial $\tilde{p}_{N+1}$ vanishes under clustering, and therefore so does $L^- \tilde{p}_{N+1}$. If this last polynomial was non-zero, it would have a total degree N, which is forbidden by corollary 1. Therefore $L^- \tilde{p}_{N+1} = 0$ and $\tilde{p}_{N+1}$ is translationally invariant.

In fact, it is possible to directly calculate $L^-\tilde{p}_i$, and one gets

$$L^-\tilde{p}_i = (N + 1 - i)\tilde{p}_{i-1}. \tag{52}$$

Note that this results only hold for $q = mN$ variables. This follows from the behaviour of $r_n^{(x)}$ under translations, which is given in appendix A.4.

Since the kernel of C_m is invariant under the action of L^i, it can be decomposed into irreducible representations of $SU(2)$. In order to prove that non-zero polynomials that vanish under the clustering have degree at least $N + 1$, it is therefore sufficient to prove it for lowest weights, that is to say translation invariant polynomials. Therefore conjecture 1 is equivalent to the following:

Conjecture 2. The only translationally invariant symmetric polynomial P in mN variables with degree N or less that vanishes under the clustering transformation is $P = 0$.

7. A possible road towards finishing the proof

As we saw in the previous section, we were able to prove that the *total* degree of a symmetric polynomial is at least $N + 1$, if the polynomial vanishes under the clustering transformation (37). However, we would like to show that the the maximum degree of any of the variables (i.e., the number of fluxes N_ϕ) is at least $N + 1$. Proving this statement turns out to be much harder than it looks at first. One of the reasons is that the clustering we consider is a non-local process. Namely, the positions of the various clusters are arbitrary. Therefore, proving that the total degree is bounded from below is already a non-trivial result. What makes proving a bound on the total degree more tractable in comparison to proving a bound on the degree, is that upon taking linear combinations, the total degree of the polynomials does not change, provided the resulting polynomial does not vanish. The degree of the polynomial, however, can be lowered by taking linear combinations.

To show that the rank of the reduced density matrix for the particle cut does indeed satisfy the upper bound given in the introduction, it suffices to prove that the clustering map $C_m : \Lambda_{mN_B}^{N_A} \longrightarrow \Lambda_{N_B}^{mN_A}$, is injective if $N_A \leqslant N_B$. In the case $N_A = N_B$, the map C_m would then actually be bijective. One possible route in trying to prove this, is to find two suitable bases for Λ_{mN}^{N} and Λ_{N}^{mN} in which the map C_m acts in an upper-triangular way, and then check that all diagonal elements are non-zero. We did not, however, succeed in finding suitable bases.

A completely different route to prove that the rank of the reduced density matrix is given by the upper bound is to try to make use of the results for the Read–Rezayi states [37]. These states are defined by the property that they vanish if $k + 1$ particles are put at the same location (in their simplest bosonic incarnation). In particular, it is known exactly that how many symmetric polynomials there are, that satisfy this clustering condition, for an arbitrary number of particles, and arbitrary degree [32, 33]. In addition, there are explicit expressions for these polynomials [33], see also [38]. Using these results, we can prove the wanted result for $N = 2$ and arbitrary m. That is, we can show that any symmetric polynomial in $2m$ variables, that vanishes if two clusters of m variables each are formed, has degree at least three.

To do so, assume that P is a polynomial in $2m$ variables that vanishes under the clustering, $C_m P = 0$. We know that the total degree of this polynomial is at least 3, and we want to show that the minimal value of the degree is three as well.

To show this, we note that the polynomial P also vanishes if we make one big cluster of $2m$ variables. From the results on the Read–Rezayi states, we know that such symmetric

polynomials have degree at least two (it vanishes, so it should vanish quadratically), and that it is unique (up to an overall factor). In addition, we know an explicit form of this polynomial P', namely

$$P'(z_1, \ldots, z_{2m}) = S[(z_1 - z_3)(z_2 - z_3)], \tag{53}$$

where S denotes the complete symmetrization over all $2m$ variables. In this case, by inspection, one can convince oneself that P' does not vanish if one makes two clusters of m variables. Thus, the minimal degree of a polynomial P in $2m$ variables that does vanish under C_m has degree at least three. In fact, it is not too hard to find an expression similar to the one for P', namely

$$P(z_1, \ldots, z_{2m}) = S[(z_1 - z_4)(z_2 - z_4)(z_3 - z_4)]. \tag{54}$$

It is not completely obvious that this vanishes under the clustering for $m > 2$, but one can convince oneself that after symmetrization, one indeed does get zero.

Though it is not going to be easy, one could try to proceed in this way. Constructing the next case, namely polynomials that vanish for three clusters of m variables, is already more involved. Writing down an explicit form similar to the ones above is not straightforward, but one can for instance symmetrize the following combination

$$\begin{aligned} P(z_1, \ldots, z_{3m}) = S\big[&(z_1 - z_5)(z_2 - z_5)(z_3 - z_5)(z_4 - z_5) \\ &- (z_1 - z_5)(z_2 - z_5)(z_2 - z_6)(z_3 - z_6) \big]. \end{aligned} \tag{55}$$

This polynomial is the unique polynomial (up to a constant factor) in $3m$ variables, of degree and total degree 4, that vanishes under formation of three clusters of m variables. We stress, however, that this alone does not imply that there are no polynomials of degree three, that vanish under the same clustering conditions.

The lowest degree polynomial for $N = 4$ and arbitrary m can still be written by symmetrizing an expression like the one in equation (55), i.e., two terms only, but it seems likely that these expressions become more complicated upon increasing N. In addition, having these explicit expressions does not help in excluding the existence of lower degree polynomials with the same clustering conditions.

8. Discussion

In this paper, we revisited the study of the PES for the $\nu = 1/m$ Laughlin states, in particular the rank of the associated reduced density matrix. To determine this rank, we make use of the rank of the reduced density matrix for the $\nu = 1$ Laughlin state. We showed that to relate the rank for the $\nu = 1/m$ Laughlin state to the case $\nu = 1$, one has to prove a bound on the degree of symmetric polynomials that vanish under the formation of certain clusters. Though we were not able to finish the proof of this statement, we made substantial progress by explicitly constructing a set of polynomials that vanish under the clustering, and we proved that the *total* degree of these polynomials is bounded from below.

We commented on a possible, though most likely rather hard, route towards finishing the proof. In this paper, we concentrated on the Laughlin states. It would be interesting to see if similar methods can be used to make progress on different model states, such as the Moore–Read and Read–Rezayi states, that exhibit excitations obeying non-Abelian statistics.

Acknowledgments

Appendix A. Properties of the polynomials $r_n^{(x)}$

In this appendix we introduce a family of symmetric polynomials $\{r_n^{(x)}\}$ defined through the generating function

$$\exp\left(-x\sum_{k=1}^{\infty}p_k\frac{t^k}{k}\right) = \sum_{n=0}^{\infty}r_n^{(x)}\frac{t^n}{n!}. \tag{A.1}$$

The key property of the $r_n^{(x)}$'s is their behaviour under the clustering transformation C_m:

$$C_m\left(r_n^{(x)}\right) = r_n^{(m\,x)}, \tag{A.2}$$

which is a direct consequence of their definition. Further properties follow from the generating function, namely

- $r_n^{(1)} = (-1)^n\,n!\,e_n$,
- $r_n^{(-1)} = n!\,h_n$ with $h_n = \sum_{\lambda\vdash n}m_\lambda$,
- $\partial_x r_n^{(x)}\big|_{x=0} = -(n-1)!\,p_n$ for $n \geqslant 1$,

where the first two relations are obtained by comparison to the generating functions for the e_n and h_n, see for instance [34]. Therefore, this family of polynomials interpolates between power sums p_n, elementary symmetric polynomials e_n, and complete homogeneous symmetric polynomials h_n. We give one additional property,

$$r_n^{(x+y)} = \sum_{k=0}^{n}\binom{n}{k}r_k^{(x)}r_{n-k}^{(y)}, \tag{A.3}$$

that follows from the definition.

A.1. Explicit formulas for $r_n^{(x)}$

The generating function can be expanded using Bell's polynomials [39], yielding an explicit expression for $r_n^{(x)}$, that is

$$r_n^{(x)} = n!\sum_{\lambda\vdash n}(-x)^{l(\lambda)}\frac{p_\lambda}{z_\lambda}. \tag{A.4}$$

Alternatively, this expression can be obtained by acting with C_x on Newton's identity expressing elementary symmetric polynomials in terms of power sums (here, we allow x to be real, and set $C_x p_i = x p_i$, for an infinite number of variables). Another explicit expression is given in terms of a determinant of power sums

$$r_n^{(x)} = (-1)^n \begin{vmatrix} xp_1 & 1 & 0 & \cdots & 0 \\ xp_2 & xp_1 & 2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ xp_{n-1} & xp_{n-2} & \cdots & xp_1 & n-1 \\ xp_n & xp_{n-1} & \cdots & xp_2 & xp_1 \end{vmatrix}. \tag{A.5}$$

The first few polynomials are given by

$$r_0^{(x)} = 1,$$
$$r_1^{(x)} = -xp_1,$$
$$r_2^{(x)} = x^2 p_1^2 - xp_2,$$
$$r_3^{(x)} = -x^3 p_1^3 + 3x^2 p_1 p_2 - 2xp_3,$$
$$r_4^{(x)} = x^4 p_1^4 - 6x^3 p_1^2 p_2 + 3x^2 p_2^2 + 8x^2 p_1 p_3 - 6xp_4,$$
$$r_5^{(x)} = -x^5 p_1^5 + 10x^4 p_1^3 p_2 - 15x^3 p_1 p_2^2 - 20x^3 p_1^2 p_3$$
$$\qquad + 20x^2 p_2 p_3 + 30x^2 p_1 p_4 - 24xp_5.$$

From triangularity it follows that the family $\{r_1^{(x)}, \ldots, r_n^{(x)}\}$ algebraically spans all symmetric polynomials in n variables as long as $x \neq 0$.

A.2. Degree of $r_n^{(x)}$

Let us consider the monomial decomposition of $r_n^{(x)}$

$$r_n^{(x)} = \sum_{\lambda \vdash n} q_n^{(\lambda)}(x) \, m_\lambda. \tag{A.6}$$

We are going to compute the first coefficient, namely $q_n^{(\lambda)}(x)$ for $\lambda = (n)$. If this first coefficient is non-zero, the degree of $r_n^{(x)}$ is n. This coefficient is a polynomial of degree n in x, since from (A.4) we have $r_n^{(x)}(z, 0, 0, \ldots) \sim (-x)^n z^n$ as x goes to infinity.

We now take $x = m$ to be an integer, and write $r_n^{(m)} = C_m(r_n^{(1)}) = (-1)^n n! \, C_m(e_n)$. By considering the definition of e_n, it follows that $m \geqslant n$ in order for $r_n^{(m)}(z, 0, 0, \ldots)$ to be non-zero, because $m_\lambda = 0$ if the number of variables is less than $l(\lambda)$.

Therefore, $q_n^{(\lambda=(n))}(x)$ vanishes for $x = 0, 1, \ldots, n-1$. It follows from the asymptotic behaviour given above that

$$q_n^{(\lambda=(n))}(x) = (-1)^n x(x-1)(x-2)\cdots(x-n+1). \tag{A.7}$$

Hence, the degree of $r_n^{(x)}$ is n as long as $x \neq 0, 1, \ldots, n-1$.

A.3. Monomial decomposition of $r_n^{(x)}$

In this appendix we quote the full monomial decomposition of $r_n^{(x)}$, namely

$$q_n^{(\lambda)}(x) = (-1)^n \binom{n}{\lambda_1, \ldots, \lambda_n} \prod_{i=0}^{n-1} (x-i)^{\lambda_{i+1}^t}, \tag{A.8}$$

where λ^t stands for the transpose of λ. The parts of λ^t are given by $\lambda_i^t = l(\lambda) - \sum_{j=1}^{i-1} n_j(\lambda)$, or equivalently, $\lambda_i^t - \lambda_{i+1}^t = n_i(\lambda)$. Below, we sketch how this can be established.

Lemma 1. *The monomial expansion of $\partial_x^j r_n^{(x)}\big|_{x=0}$ involves only partitions λ such that $\lambda_{j+1} \leqslant 0$, i.e., $\lambda_i = 0$ for $i \geqslant j + 1$.*

Proof. From the generating function for the $r_n^{(x)}$ we get

$$\partial_x^j r_n^{(x)}\big|_{x=0} = (-1)^j n! \sum_{\substack{k_1,\ldots,k_j \geqslant 1 \\ k_1 + \cdots + k_j = n}} \prod_i \frac{p_{k_i}}{k_i}. \tag{A.9}$$

In this expression each term $\prod_i p_{k_i}$ has a monomial decomposition that involve only partitions λ with a length $l(\lambda) \leqslant j$, from which lemma 1 follows. ∎

Lemma 2. *The monomial expansion of $\partial_x^j r_n^{(x)}\big|_{x=i}$, with i an integer, involves only partitions with $\lambda_{j+1} \leqslant i$.*

Proof. The case $i = 0$ boils down to lemma 1. Lemma 2 can be proven by induction on i using

$$\partial_x^j r_n^{(x)}\big|_{x=i} = n! \sum_p \frac{(-1)^{n-p}}{p!} e_{n-p} \left(\partial_x^j r_p^{(x)}\big|_{x=i-1}\right), \tag{A.10}$$

which follows from taking ∂_x^j in (A.3), namely

$$\partial_x^j r_n^{(x+y)} = \sum_p \binom{n}{p} \partial_x^j r_p^{(x)} r_{n-p}^{(y)}, \tag{A.11}$$

and then choosing $x = i - 1$ and $y = 1$. ∎

Corollary 3. *The coefficient $q_n^{(\lambda)}(x)$ is of the form*

$$q_n^{(\lambda)}(x) = c_n^{(\lambda)} \prod_i (x - i)^{\lambda_{i+1}^t}. \tag{A.12}$$

Proof. Lemma 2 is equivalent to stating that $\partial_x^j q_n^{(\lambda)}(x) = 0$ for $x = 0, 1, \ldots, \lambda_{j+1} - 1$. Thus, $x = i$ is a root with degeneracy λ_{i+1}^t of the polynomial in x $q_n^{(\lambda)}(x)$, because λ_{i+1}^t is the number of parts of λ that are bigger or equal to $i + 1$. Since this is true for all $i \in \mathbb{N}$, we have a total of $\sum_i \lambda_i^t = n$ zeros. Since $q_n^{(\lambda)}(x)$ is of degree at most n, corollary 3 follows. ∎

Lemma 3. *The coefficients $c_n^{(\lambda)}$ are given by*

$$c_n^{(\lambda)} = (-1)^n \frac{n!}{\lambda_1! \cdots \lambda_n!} = (-1)^n \binom{n}{\lambda_1, \ldots, \lambda_n}. \tag{A.13}$$

Proof. The asymptotic behaviour of (A.6) for x going to infinity yields

$$(-1)^n p_1^n = \sum_{\lambda \vdash n} c_n^{(\lambda)} m_\lambda. \tag{A.14}$$

Lemma 3 follows by expanding the lhs using the multinomial theorem (assuming the number of variables $p \geqslant n$)

$$\left(\sum_{i=1}^{p} x_i\right)^n = \sum_{\substack{k_1,\ldots,k_p \geqslant 0 \\ k_1+\cdots+k_p=n}} \frac{n!}{k_1! \cdots k_p!} \prod_i x_i^{k_i}, \tag{A.15}$$

and then gathering the terms of the rhs into symmetric monomials. ∎

A.4. Behaviour of $r_n^{(x)}$ under translations

Translations are well defined in the case of finitely many variables $\{x_1, \ldots, x_r\}$, in which case we set $p_0 = r$, i.e., the number of variables. In that case L^- is the generator of translations

$$L^- = \sum_{i=1}^{r} \frac{\partial}{\partial x_i}. \tag{A.16}$$

By Leibniz's rule, its action on $p_\lambda = \prod_j p_{\lambda_j}$ is

$$L^- p_\lambda = \sum_{j \geqslant 1} n_j(\lambda) j p_{j-1} p_{\lambda \setminus \{j\}}, \qquad p_0 = r, \tag{A.17}$$

where the $n_j(\lambda)$ is the number of parts of λ that equal j, and $\lambda \setminus \{j\}$ denotes the partition derived from λ by deleting one part that equals j. We can now act on $r_n^{(x)}$,

$$L^- r_n^{(x)} = n! \sum_{\lambda \vdash n} \frac{(-x)^{l(\lambda)}}{z_\lambda} \sum_{j=1}^{n} n_j(\lambda) j p_{j-1} p_{\lambda \setminus \{j\}}. \tag{A.18}$$

We can change the summation variable from λ to $\mu = \lambda \setminus \{j\}$, after noticing that μ is in one-to-one mapping with (λ, j), since

$$j = n - |\mu|, \qquad \lambda = \mu \bigcup \{j\}. \tag{A.19}$$

In particular $l(\lambda) = l(\mu) + 1$, $n_j(\lambda) = n_j(\mu) + 1$ and $z_\lambda = z_\mu(n_j(\mu) + 1)j$. We find

$$L^- r_n^{(x)} = n! \sum_{\mu \vdash 0}^{n-1} \frac{(-x)^{l(\mu)+1}}{z_\mu} p_{n-1-|\mu|} p_\mu. \tag{A.20}$$

We now split the sum into two parts. First the term $|\mu| = n - 1$ is simply

$$n! \sum_{\mu \vdash n-1} \frac{(-x)^{l(\mu)+1}}{z_\mu} p_0 p_\mu = -x n p_0 r_{n-1}^{(x)}. \tag{A.21}$$

Now the reminder is

$$n! \sum_{\mu \vdash 0}^{n-2} \frac{(-x)^{l(\mu)+1}}{z_\mu} p_{n-1-|\mu|} p_\mu = n(n-1) r_{n-1}^{(x)}, \tag{A.22}$$

as can be seen from the determinant formula (A.5) by repeatedly developing along the last column until the matrix is 1×1. These last 'determinants' correspond to the factor $p_{n-1-|\mu|}$ in the sum above. Finally we get

$$L^- r_n^{(x)} = n\left(n - 1 - x p_0\right) r_{n-1}^{(x)}, \tag{A.23}$$

from which equation (52) in the main text follows.

Appendix B. An alternate set of generators

In this appendix, we briefly introduce an alternate set of generators p_i', that could be used in the proof in section 5. This set of generators of $\Lambda_{q=mN}$ is constructed to satisfy $p_i' = p_i$ for $i \leqslant N$, and $C_m(p_i') = 0$ for $i > N$, and have total degree i. In the construction, to keep track of the number of variables of the power sums, we introduce an additional index N, so

$$p_{i,N}(x_1, ..., x_N) = x_1^i + \cdots + x_N^i. \tag{B.1}$$

Recall that $\mathcal{A}_N = \{p_{1,N}, ..., p_{N,N}\}$ are algebraically independent and generates all symmetric polynomials in N variables. In particular for each $i \in \mathbb{N}^*$ there exists a unique polynomial $T_{i,m,N}$ in N variables, such that

$$m\, p_{i,N} = T_{i,m,N}\left(m\, p_{1,N}, ..., m\, p_{N,N}\right). \tag{B.2}$$

The $p_{i,q}'$ are now defined as follows

$$p_{i,q}' = \begin{cases} p_{i,q}, & 1 \leqslant i \leqslant N, \\ p_{i,q} - T_{i,m,N}\left(p_{1,q}, ..., p_{N,q}\right), & N < i \leqslant q. \end{cases} \tag{B.3}$$

By construction, they obey $C_m(p_{i,q}') = 0$ for $i \geqslant N + 1$, are non-zero, and have total degree i. In addition, they form an alternate generating set $\mathcal{A}_q' = \{p_{1,q}', ..., p_{q,q}'\}$ of Λ_q, because the $p_{i,q}$ of the generating set $\mathcal{A}_q$ can be expressed in terms of the $p_{i,q}'$ in equation (B.3).

Since, as is shown in corollary 1, $\tilde{p}_{N+1}$ is the unique (up to a scale factor) polynomial that vanishes under the clustering C_m, we find that $p_{N+1}' \propto \tilde{p}_{N+1}$, and that $L^- p_{N+1}' = 0$. In addition, by a direct calculation, one finds that $L^- p_{i,q}' = i p_{i-1,q}'$ for $1 \leqslant i < N$ and $N + 1 < i \leqslant q$.

Namely, by acting on both sides of the definition of $T_{i,m,N}$, equation (B.2) with L^- gives

$$\begin{aligned} L^- T_{i,m,N}\left(mp_{1,N}, ..., mp_{N,N}\right) &= q T_{i,m,N;1}\left(mp_{1,N}, ..., mp_{N,N}\right) \\ &\quad + \sum_{j=2}^{N} m\, j\, p_{j-1,N}\, T_{i,m,N;j}\left(mp_{1,N}, ..., mp_{N,N}\right) \\ &= i m p_{i-1,N} = i T_{i-1,m,N}\left(mp_{1,N}, ..., mp_{N,N}\right), \end{aligned}$$

where $T_{i,m,N;j}$ denotes the derivative of $T_{i,m,N}$ with respect to its jth argument. In particular by setting $X_j = mp_{j,N}$, we find that for general arguments,

$$i T_{i-1,m,N}\left(X_1, ..., X_N\right) = q T_{i,m,N;1}\left(X_1, ..., X_N\right) + \sum_{j=2}^{N} j X_{j-1} T_{i,m,N;j}\left(X_1, ..., X_N\right). \tag{B.4}$$

We can now act with L^- on both sides of the definition of $p_{i,q}'$, equation (B.3). Using the relation equation (B.4), we find that, for $i > N + 1$,

$$\begin{aligned} L^- p_{i,q}' &= L^- p_{i,q} - L^- T_i\left(p_{1,q}, ..., p_{N,q}\right) \\ &= i p_{i-1,q} - i T_{i-1}\left(p_{1,q}, ..., p_{N,q}\right) \\ &= i p_{i-1,q}', \end{aligned} \tag{B.5}$$

which is what we wanted to show. We note that in the case that $i = N + 1$, the only thing that changes in the argument above is that the lhs of equation (B.4) is replaced by iX_N, leading to the result $L^- p'_{N+1,q} = 0$, which we showed in the main text using a different method.

Finally, we mention that it is also possible to prove that the degree of $p'_{i,q}$ equals i for $N + 1 \leqslant i \leqslant 2N + 1$, directly from the definition. We believe that the degree of $p'_{i,q}$ also equals i for $2N + 1 < i \leqslant q$, but did not find a proof of this.

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