Generation of magnetic fields on galactic scales

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Preface

In these pages we will go through the topic of astrophysical magnetic fields, focusing on galactic fields, their observation and the theories that have been developed for a proper understanding of the these physical phenomena. We review the main work in the study of galactic magnetic fields, often seeing how it is important to deal with problems of general validity in order to be able to point out the right elements needed for a correct interpretation of specific situations. We also aim to summarize some of the conflicts that arise using different theoretical approaches to be proficient in future choices of our research guidelines. This thesis consists in an introductory text and three papers dealing with some specific topics that are introduced in the first three chapters.

In the first chapter we will talk about the state of the art of the observations of galactic fields. We review current techniques and observations. In the second chapter we describe the current theories that best describe the generation of magnetic fields. We also mention here two of the three works presented in this licentiate thesis. We will then deal with the possibility to have a proper measure of the $\alpha$ effect in numerical simulations of dynamo action. Then we consider a particular aspect of magnetic helicity, that is, its connection with the topology of the magnetic field in a given system. In the third chapter we focus on theories related to galactic fields and their validity. We also present our work on the generation of vorticity in the interstellar medium as well as a study of turbulent diffusivity in a system presenting spherical expansion waves.
List of papers included in the thesis


Conference papers not included in the thesis


Summary of the papers

This work is a computational and analytical study of some general features of fluid motions and magnetic fields that are applied in the context of galactic dynamics. Magnetic field generation is generally the result of gas motions in conductive media. Consequently we begin studying some aspects of motion’s nature in the interstellar medium (ISM).

In Paper I, we argue that the driving of turbulence in galaxies is essentially irrotational. However, in a systematic investigation of the mechanisms leading to a conversion of irrotational to vortical motions we are for the first time able to quantify the relative importance of the various mechanisms. The presence of vorticity is a necessary condition for having helicity, which is proportional to the dot product of vorticity and velocity and can lead to an $\alpha$ effect. This is a highly controversial subject, partly because it is so hard to quantify the value of $\alpha$ for a given flow. Therefore, we study in Paper II two of the main methods for determining $\alpha$ computationally— the imposed-field method and the test-field method. It is now well known that the $\alpha$ effect produces magnetic helicity, which characterizes the mutual linkage of magnetic flux and that is a conserved quantity in flows with large magnetic Reynolds number. This picture helps understanding that the violation of magnetic helicity conservation is connected with the difficulty of breaking interlocked flux structures apart. However, in Paper III it is shown that this interpretation is too naïve. In particular, using two similar flux configurations of triply interlocked flux, it is shown that the relative field orientation also matters, because in one case there is no net helicity while in the other there is.

This work has already been reported at various international conferences and will be published in various proceedings; see papers 1–7, that are not included in this thesis.
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# Preface

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Chapter 1

Magnetic fields in a galactic environment

1.1 Magnetic fields on different scales

It is known that almost all astrophysical bodies are magnetized: planets, stars, the interstellar medium (hereafter ISM), galaxies and clusters of galaxies. These two last objects are currently the only ones where a large scale magnetic field can be seen inside the body itself [Brandenburg and Subramanian, 2005]. The observations of such fields lead to a natural question: how are these fields generated and how do they evolve? We shall see that for the time being some theories have been proposed but none of them is able to fully explain all the observed phenomena.

The first astrophysical magnetic field to be observed has been that of our planet, the Earth. Its magnetic flux density reaches $0.6 \, \text{G}$ in proximity of the poles. After that, at the beginning of the 20th century, G. E. Hale interpreted a particular split in the spectral lines coming from sunspots as due to magnetic field through the Zeeman effect. It is now known that in a sunspot there are magnetic fields of the order of $10^3 \, \text{G}$. Magnetic fields then have been observed over the years on other planets of the solar system too, with magnitudes ranging from $10^{-4} \, \text{G}$ up to $10 \, \text{G}$. Speaking instead about cosmic objects located outside the solar system, many types of stars have been shown to harbour a magnetic field, going from flux densities of $10^4 \, \text{G}$ for hot stars (spectral type A and B) to $10^8 \, \text{G}$ for white dwarfs and $10^{12} \, \text{G}$ for neutron stars. When speaking about galaxies we are dealing with fields presenting average fluxes of $10^{-5} \, \text{G}$, due mainly to the huge typical dimension of these objects. We will see how their structure is rather complex but still presents both global and local symmetries.
We start then reviewing the state of the art of the observation of galactic fields and we will then move our attention to the proposed theories, different approaches and possible ways to be followed in order to achieve a better comprehension of magnetic fields.

1.2 How to observe galactic magnetic fields

To speak about the first observation of galactic magnetic fields we need to go back to the late 1940ies: the idea of magnetic fields in the galaxy came up when polarized optical emission suggested a structure resembling that expected of magnetic field lines. The first observations of such a phenomenon has been performed in 1949 by [Hiltner, 1949] and [Hall, 1949]. In fact, interstellar dust grains align in an external magnetic field and emit polarized infrared radiation. This polarized emission is then mainly associated with dust grains that can line up with the ambient magnetic field, as already pointed out in the same year [Davis and Greenstein, 1949].

1.2.1 Zeeman splitting

Another way to observe magnetic fields has been, historically speaking, to measure the splitting of spectral lines due to the Zeeman effect. This is a quantum mechanics effect: atomic energy levels are usually degenerate with respect to the total angular momentum direction. When an atom is instead in a magnetic field $B$, those levels can split because the atom acquires an additional energy that depends on the mutual orientation of the angular momentum and the external magnetic field. Without giving any technical detail here, we just report the final formula of the so-called normal Zeeman effect, that is the splitting of one line $\nu_0$ into two additional ones, $\nu_{\pi}, \nu_\sigma$

$$\nu_\sigma = \nu_0 \pm \frac{g \mu}{\hbar} B, \quad \nu_\pi = \nu_0,$$  \hspace{1cm} (1.1)

in which $\nu_0$ is the basic frequency, $g$ is the Landé factor, that is a factor giving the degeneration of an energy level in terms of orbital and spin momenta, $\mu$ is the Bohr magneton and $\hbar$ is the Planck constant. An anomalous Zeeman effect, as well as the normal one, can happen: in this case the Landé factor of the upper level is different from that of the lower one.

However, the Zeeman effect is useful only for objects dense enough and with a rather strong magnetic field in order to allow the line splitting to be visible. It is rather common that the thermal Doppler effect produces a broadening much bigger than the Zeeman effect. For example the 21 cm
1.3 Magnetic fields in spiral galaxies

The line of neutral hydrogen suffers a Zeeman broadening of about 30 Hz when embedded in a typical galactic field of the order of $10^{-5}$ G, that is comparable with the thermal Doppler broadening corresponding to a temperature of 1 K, that is two orders of magnitude less than the typical temperature of the ISM. Consequently, the Zeeman effect of lines like the 21 cm line of neutral hydrogen, the 18 cm line of OH and some other molecular lines of CO and CN can be observed in star forming regions (typically $n = 10^5 - 10^6$ cm$^{-3}$ and $B \simeq 1$ m G [Shukurov and Sokoloff, 2008]). In this way it has been found a $20 \mu$ G field in the Perseus spiral arm [Verschuur, 1968]. Zeeman effect is instead very important for observing solar and stellar magnetic field.

1.2.2 Faraday rotation and synchrotron emission

Radio observations of galactic synchrotron emission is nowadays the main way to observe their magnetic fields [Brandenburg and Subramanian, 2005]. This emission is due to electrons, or charged particles, moving along magnetic field lines. They move following a spiral pattern, then producing a polarized emission. When immersed in a large-scale field, i.e. a rather regular field, the polarization enhances, while non-ordered fields bring a decrease of it. What is really searched for, in order to observe interstellar magnetic fields, is the so-called Faraday rotation. This consists in a weighted integral of magnetic field along the line of sight between the observer and a background source. The result of such an integral is then an average measure of the magnetic field over that path.

1.3 Magnetic fields in spiral galaxies

In nearby spiral galaxies the average total field that is obtained from total synchrotron intensity ranges from $4 \mu G$ in M31 up to about $15 \mu G$ in M51: the mean value is $B = 9 \mu G$ for a sample of 74 galaxies. The typical degree of polarization of synchrotron emission from galaxies at short radio wavelengths is $p = 10 - 20\%$: from this, considering also the limited resolution of the observations, one can obtain the ratio $\langle B \rangle / B = 0.6 - 0.7$ [Shukurov and Sokoloff, 2008]. The total value of the equipartition field in the solar neighborhood is $B = 6 \pm 2 \mu G$. This is obtained from the synchrotron intensity of the diffuse galactic background: using these two last values it can be argued that the local regular field $B$ has a strength of $B = 4 \pm 1 \mu G$, while for the random component of the total field we have $b = (B^2 - \langle B^2 \rangle)^{1/2} = 5 \pm 2 \mu G$ [Shukurov and Sokoloff, 2008]. If we look at specific cases we have, for example, as values for the average equipartition field, $4 \mu G$ for M33, $12 \mu G$ for NGC 6946,
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and 19\(\mu G\) for NGC 2276 \cite{Zweibel97}. From such data, as well as from Faraday rotation measures, we can see how the random component of the field is bigger than the regular one. It has to be pointed out the existence of a discrepancy between the \(\overline{B}\) observed in the Milky Way and the one measured for other spiral galaxies. In fact, for our galaxy we observe \(\overline{B} = 1 - 2\mu G\), that is, a lower value than the aforementioned ones. There could be several explanations for this problem, as showed for example by \cite{Beck03} and \cite{Sokoloff98}. One of these could be the difference in depth probed by the total synchrotron emission and Faraday rotation measures in observations of extragalactic and galactic sources.

The study of individual magnetic field structures and their relative fluctuations tell us that on scales of the order of 100 pc magnetized interstellar shells are observed, produced by single or clustered supernovae \cite{Zweibel97}. This is evident for nearby objects like the North Polar Spur \cite{Egger95}. In these shells the magnetic field is enhanced by compression.

We can then summarize the observations of magnetic fields of spiral galaxies. The total strength of the field is \(B \simeq 5-12\mu G\), while the global magnetic field is \(\overline{B} \simeq 3 - 7\mu G\) and the ratio of energy densities in random and regular magnetic fields is \(\langle b^2 \rangle / B^2 \simeq 3\) \cite{Shukurov08}. In the Milky Way a magnetic field with a global quadrupolar parity has been observed, while this has not yet been observed elsewhere \cite{Frick01}. The global pattern of the field is that of a spiral, similar to the spiral arms, but there is also a huge variety of structures, like magnetic arms and field reversals between discs and halo \cite{Fletcher10}. We can observe one of these reversal zones close to the solar system and in fact the strength of the field nearby the Sun, that is \(B_\odot \simeq 2\mu G\), is not representative of the one of spiral galaxies. The pitch angle of the spiral pattern is characterized by a pitch angle given by \(p_B = \arctan (\overline{B}_r / \overline{B}_\phi) = -(10-30)\).

In barred galaxies the global configuration of the magnetic field is instead expected to be different from that of spiral galaxies. Interstellar magnetic field are in fact strongly affected by the non-axisymmetric gas flow and large scale shocks. In particular the regular magnetic field might be enhanced by velocity gradients, while the dynamo action would be influenced by the presence of a bar \cite{Beck05}.

Regarding magnetic fields in halos, their scale height is typically 4 kpc, i.e. much larger than the density scale height of about 70 pc. It has been speculated that even the halo may be turbulent, although the source of turbulence is not clear. A magneto-rotational instability is a possibility that has been discussed in connection with the outskirts of galaxies where supernova driving cannot be invoked. In-situ dynamo action in galactic halos
1.4 Magnetic fields in dwarf and irregular galaxies

Dwarf galaxies are the most numerous species of galaxies in the universe. Nevertheless they are very difficult to observe being very weak objects especially in the radio domain. As a consequence, it is not well known how easy and common is the generation of magnetic fields in these galaxies. Recently, thanks to investigation of the radio emission of nearby dwarf galaxies [Chyży, 2010], a trend has been observed that indicates that dwarf galaxies seem to have predominately weak magnetic fields, with strength of about 4 $\mu$G, that is about three times smaller than in normal spirals. On the other hand, in 2000, a strong polarized emission was discovered in an optically bright dwarf galaxy, NGC 4449. In this case the strength of the total magnetic field is of about 12 $\mu$G, while the regular component is about 8 $\mu$G [Chyży, 2010]. In general it is found that magnetic fields depend on the surface density through the galactic star formation rate. The low mass of irregular dwarf galaxies is the main cause for their interstellar medium to be particularly vulnerable to disruption by intense episodes of star formation. Recent important studies have been conducted to investigate the role of the magnetic field in the interstellar medium of post-starburst dwarf galaxies, like the one by [Kepley et al., 2010] about NGC 1569. This is a nearby dwarf irregular galaxy which shows an intense starburst of 10–40 Myr ago and it has then been studied with observations at 20 cm, 13 cm, 6 cm and 3 cm [Kepley et al., 2010]. This investigation has shown the presence of a strong polarized emission (3 cm and 6 cm) as well as a weak one (13 cm and 20 cm). The main importance of these observations is that of deriving the strength of the magnetic fields and to compare the magnetic pressure with that of other components in the interstellar medium. In the case of NGC 1569 it has been calculated [Kepley et al., 2010] that the total magnetic field strength is about 38 $\mu$G in the central regions and 10–15 $\mu$G in the halo. In the center of the galaxy the uniform component of the field is of the order of 3–9 $\mu$G, while in the halo stronger: in any case the random component of the field is the predominant one. With those data it is found that the magnetic pressure is of the same order of magnitude, or less, than the other components in NGC 1569. The 20 cm investigation has also confirmed the idea that an extended radio continuum halo is present. Dwarf galaxies with extended radio continuum halos
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Among the irregular galaxies whose fields have been monitored we mention here NGC 4449 [Chyży et al., 2000], IC 10 and NGC 6822 [Chyży et al., 2003]. In the case of NGC 4449 a strong regular field has been found. This is surprising, since the structure of the galaxy itself is lacking an ordered rotation pattern that was expected to be necessary to for dynamo action. Nevertheless, rotation could also act on a smaller scale to help the dynamo process to take place. On the other hand it is common to have, in galaxies with a more regular structure, a stronger random component of the field. Even in spiral galaxies it is know that the spiral pattern is followed in the magnetic field, but the random component is usually of the same order of magnitude as the regular one. The aforementioned values of NGC 4449 are comparable with those related to radio-bright spirals. One example of a magnetic field configuration in a spiral galaxy is shown in Fig. 1.4. In IC 10 (Fig. 1.3) the field is mostly random, reaching a strength of 14 $\mu$G. The regular component is about 3 $\mu$G.

Regarding high-redshift dwarf galaxies, they could have a higher star
1.4 Magnetic fields in dwarf and irregular galaxies

Figure 1.2: The distribution of Faraday rotation measures in the disk of NGC 4449 between 8.44 and 4.86 GHz (figure adapted from Chyży et al., 2000)

Figure 1.3: $H_\alpha$ image of IC 10 (figure adapted from Chyży et al., 2003)
Figure 1.4: Magnetic fields mapped on an optical image of M 51 (figure adapted from [Fletcher et al., 2010]). The image shows the contours of the total radio intensity and polarization vectors at 6cm wavelength, combined from radio observations. The magnetic field seems to follow rather well the optical spiral structure. However, also the regions between the spiral arms contain strong and ordered fields.
formation rate and, consequently, also a stronger magnetic field. However, such distant dwarfs might be quite different from the local ones and then a comparison could be difficult.

1.5 Magnetic fields in high redshift galaxies

The study of magnetic fields in high-redshift galaxies is important mainly for understanding how those fields evolved in time. Indeed the origin of magnetic fields in galaxies is still enigmatic and ideas like that of dynamos from supernovae-driven turbulence are not able to provide a complete understanding of observations. In order to observe magnetic fields in a distant spiral galaxy one needs to find at least one polarized background source to perform a Faraday rotation measurement, that is a measure of the angle formed by the polarization vector and the field. This is of course not that obvious for distant galaxies, mainly due to their small angular size. In fact, what happens for intermediate redshift galaxies is that they can occasionally lay on the line of sight of some distant quasar. In such a case a magnetic field in the galaxy might then be revealed. What is nowadays done to approach the study of magnetic fields in distant spiral galaxies is summarized in the work of [Stil, 2009]: the polarization of unresolved fields in spiral galaxies is analyzed to obtain statistical informations on the uniformity of magnetic fields and Faraday depolarization in galaxies. This is a work in progress: when the studied galaxies will form a statistically significant sample, then it will be possible to use this sample for studying distant galaxies. It was recently demonstrated [Bernet et al., 2008] that magnetic field strengths of distant quasars, as observed by Faraday Rotation Measures, are comparable to those seen today. To determine whether these fields belong to the quasar or were distributed along the line of sight, Mg II absorption has been studied. These lines are associated with large rotation measures. This absorption is a phenomenon occurring in halos of galaxies and then it is unavoidable to associate their existence with the presence of magnetic fields in halos. Quasars up to $z \approx 3$ have been observed [Bernet et al., 2008]. In the same work it is pointed out how, at high-redshift, Faraday rotation is enhanced with respect to low redshift sources. They find how, for $z \approx 1.3$, magnetic fields had a strength comparable with that observed in todays galaxies. Indeed, having Mg II as a probe of the redshift at which Faraday Rotation is produced, they conclude that high redshift emission passes through more Mg II zones (like halos) before reaching us and then they present a stronger Faraday rotation.
2
Generation of magnetic fields

2.1 What is a dynamo?

The main idea of dynamo theory is that a magnetic field can be amplified through self-excitation. Magnetic fields exist either through permanent magnetization or through electric currents. In the first case we have a stationary magnetic field but the stationarity is a rarely observed feature in astrophysical objects. This leads one to believe that motions of charged particles are mainly responsible for astrophysical fields, opening then the door to the problem of generation and sustainment of those currents and of the magnetic fields inducted by them.

Ernst Werner von Siemens (1816–1892) was the first, in 1866, in proposing the idea that a conducting matter can possibly carry electric currents when in motion so amplifying pre-existent fields. This is then the origin of the so-called dynamo theory: a dynamo is then a process that transforms kinetic energy into magnetic energy. Such a process is predicted and described by the so-called induction equation

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta \nabla^2 B,$$

(2.1)

that can be derived directly from Maxwell equations. Here $\eta$ is a diffusion coefficient. This equation clearly shows how the time evolution of a magnetic field depends on the velocity of the medium as well as on the magnetic field itself. It also tells us that when having a null magnetic field initially, this cannot lead to any production of magnetic field by induction. We are then dealing with a process of amplification of a field that is completely different from the one of generation.
In its first formulation, the currents needed to self excite a magnetic fields have been thought to be carried by a solid conducting body, for example a rigid disc. When such a disc is dipped in a magnetic field and is moving, for example rotating around its axis, an electromotive force can occur in the disc, giving birth to a difference of potential between the edge and the axis of the disc. To produce a current it is then only necessary to connect the edge and the axis, using for instance a conducting wire. We are then able to transform the kinetic energy into electrical current and then generate a magnetic field that can sustain the generation of the aforementioned electromotive force, being then a self-sustained field.

However astrophysical objects are not thought to be rigid bodies, but rather to be consist of plasmas and conductive fluids. As a consequence, such a simple and schematic system is not going to work, but it remains of basic importance in suggesting that a similar theory can be applied to fluids. The connection is rather straightforward when speaking about fluids in which the magnetic field is frozen-in. In this kind of fluid the identification between a field and a fluid line is possible. Nevertheless there are several astrophysical phenomena in which that condition is violated, leading thus to magnetic reconnection: a field line loses the identification with a fluid line. Moreover, motions in fluids, as well as magnetic fields, are subject to diffusion due to the action of both microscopical diffusion processes and turbulence. When speaking about dynamo action in fluids, $\eta$ in equation (2.1) indicates the micro-physical magnetic diffusivity of the fluid, usually assumed uniform.

The rest of the chapter is then dedicated to describing theories for dynamo processes in the magnetohydrodynamic regime. We will focus on the features that are most relevant for our study, like turbulence, magnetic helicity and basic computational tools that are necessary for a proper understanding of astrophysical magnetic fields.

### 2.2 Mean field theory and dynamo action

A proper understanding of dynamo process requires both physical insight and a theoretical framework in order to describe the magneto-hydrodynamical (MHD) context in which the phenomena occur. The mean-field theory [Parker, 1955a], [Moffatt, 1978], [Krause and Rädler, 1980]) is the most common theoretical approach to MHD dynamos. The main idea of mean-field theory (MFT) is that the study of turbulent systems, of which MHD dynamos are an example, can follow a two-scale approach, where velocities and magnetic fields are decomposed into mean and fluctuating components: $\mathbf{U} = \overline{\mathbf{U}} + \mathbf{u}$ and $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{b}$. The mean parts $\overline{\mathbf{U}}$ and $\overline{\mathbf{B}}$ generally vary
slowly both in space and time, and capture the global, and often the more prominent behavior of the system. The fluctuating fields on the other hand describe irregular, often chaotic small-scale effects. We have already seen how, in the case of galactic magnetic fields, observations provide values for both these components of $B$ and such a division results to be rather natural in a physical environment like galaxies.

As pointed out in paragraph (2.1), the general evolution of a magnetic field $B$ is described by the induction equation (2.1). Using the aforementioned decomposition, the induction equation can be rewritten as a set of two equations for mean and fluctuating quantities,

$$\begin{align*}
\frac{\partial B}{\partial t} &= \nabla \times (\mathbf{U} \times B) + \nabla \times \mathbf{E} + \eta \nabla^2 B, \\
\frac{\partial b}{\partial t} &= \nabla \times (\mathbf{U} \times b) + \nabla \times (\mathbf{u} \times B) + \nabla \times (\mathbf{u} \times b)' + \eta \nabla^2 b,
\end{align*}$$

(2.2) (2.3)

where $\mathbf{E} \equiv \mathbf{u} \times \mathbf{b}$ is the mean electromotive force, and $(\mathbf{u} \times \mathbf{b})' = \mathbf{u} \times \mathbf{b} - \mathbf{u} \times \mathbf{b}$. In such a description one needs to write $\mathbf{E}$ in terms of the mean field $\overline{B}$. To obtain the desired relation we can consider underlying symmetries that constrain the form of this relation. Let us take the example of a homogeneous system and assume that the turbulence is isotropic. In such a condition the vector $\mathbf{E}$ that can have constituents pointing along the mean magnetic field $\overline{B}$ and the mean current density $\overline{\mathbf{J}} = \nabla \times \overline{B}/\mu_0$ (as well as higher order spatial and time derivatives), leading to approximations such as

$$\mathbf{E} = \alpha \overline{B} - \eta_t \mu_0 \overline{\mathbf{J}}.$$  

(2.4)

The coefficients linking correlations to mean quantities are named mean-field transport coefficients, with each one describing a distinct physical effect. In equation (2.4), $\alpha$ describes the so called $\alpha$ effect, as we will see later, while $\eta_t$ quantifies the turbulent diffusion of the mean magnetic field and is called turbulent diffusivity, and $\mu_0$ is the vacuum permeability.

Equation (2.3) contains terms that can sometimes be neglected. When we are dealing with the case of fluids with small magnetic Reynolds number, that is $\text{Re}_M = UL/\eta \ll 1$, or low Strouhal number $\text{St} = U\tau_c/L \ll 1$ (where $\tau_c$ indicates a characteristic turbulent correlation time), we can drop $(\mathbf{u} \times \mathbf{b})'$ in equation (2.3) and can thus make an analytical calculation of the transport coefficients feasible. Under this approximation, known as SOCA (Second Order Correlation Approximation) or FOSA (First Order Smoothing Approximation), equation (2.3) takes the form

$$\frac{\partial b}{\partial t} = \nabla \times (\overline{\mathbf{U}} \times \overline{\mathbf{b}}) + \nabla \times (\mathbf{u} \times \overline{\mathbf{B}}) + \eta \nabla^2 \mathbf{b}.$$  

(2.5)
Figure 2.1: Dependence of $\alpha$ on $Re_M$ for the laminar Roberts flow (left) and a turbulent flow (right). Adapted from [Brandenburg et al., 2008] and [Sur et al., 2008], respectively.

In the limit of high $Re_M$ (hence small St) the coefficients $\alpha$ and $\eta_t$ reduce then to \[\alpha = -\frac{\tau_c}{3} \mathbf{u} \cdot (\nabla \times \mathbf{u}), \quad \eta_t = \frac{\tau_c}{3} \mathbf{u}^2.\] (2.6)

Equation (2.6) applies to the case of a turbulent flow where $\tau_c$ is its correlation time. In many cases of laminar flows, on the other hand, $\alpha$ and $\eta_t$ may actually decrease with increasing Reynolds number. This was first shown by [Kraichnan, 1979], who found that $\alpha$ decreases with increasing magnetic Reynolds number like $Re_M^{-1/2}$. This striking difference is demonstrated in Fig. 2.1 by comparing results obtained for the laminar Roberts flow and helically driven turbulence.

Various types of criticism to dynamo theory have been made, both related to the general idea of a dynamo and to the mean field approach. For example [Piddington, 1970, Piddington, 1981] criticized the validity of the concept of turbulent diffusion. Although there were attempts to refuse these proposals [Parker, 1973], problems remained until simulations of two-dimensional turbulence showed that there was indeed a problem [Cattaneo and Vainshtein, 1991], and that even the $\alpha$ effect may not work, but it might be quenched [Vainshtein and Cattaneo, 1992]. However, these problems were later understood to be a consequence of conservation laws (conservation of $A^2$ in two dimensions and of $A \cdot B$ in three dimensions, where $B = \nabla \times A$ is the magnetic field expressed in terms of the vector potential). Indeed when magnetic helicity is not conserved the $\alpha$ effect is not quenched.
There is a substantial underlying problem in the conflict among dynamo and non-dynamo theory. In fact dynamo theory is the only actual one that is able to answer question related to time-scale of changes in observed astrophysical fields. Moreover, numerical simulations nowadays can show how a dynamo is generated but they still remain far away from reproduce the astrophysical condition in which magnetic fields are generated. From the experimental point of view, several attempts have been done some of which have been able to generate a dynamo, like Riga and Karlsruhe experiments. Some other facilities are now on their way to be ended and a new generation of dynamo plasma experiments might soon start with the Madison experiment.

2.3 Numerical simulations: test-field method

When we talk about mean field theory we are basically dealing with the problem of how to obtain the transport coefficients. Approaching this general problem through numerical simulations is a common way to find them out, though we are talking about techniques that are still on their way to be fully developed. Direct Numerical Simulations (DNS) offer a good way to obtain these coefficients and they avoid the restricting approximations that are unavoidable in the analytical approach and that lead to deal with situations that are far away from the real ones. The simplest way to accomplish such a direct measurement is the so-called imposed field method: in the DNS an imposed large-scale field is added and its influence on the fluctuations of magnetic field and velocity is utilized to infer some of the full set of transport coefficients [Rheinhardt and Brandenburg, 2010].

A different tool, more universal in its use, is the test field method [Schrinner et al., 2005, Schrinner et al., 2007]: in a single DNS it allows to determine all the wanted transport coefficients. This method could be thus summarized: a fluctuating velocity field is taken from a DNS and inserted into a properly tailored set of equations named test equations. Their solutions, the test solutions, give the response of chosen mean fields to the interaction of with the fluctuating velocity field, that is a fluctuating magnetic field. The chosen mean fields are called test fields [Rheinhardt and Brandenburg, 2010]. This method has been applied to several models, like the ones with homogeneous turbulence with helicity [Sur et al., 2008], with shear and no helicity [Brandenburg et al., 2008] and with both of them [Mitra et al., 2009], as well as to the study of magnetorotational instability [Gressel, 2010]. This means that this method is actually able to cover several astrophysical situations, being able to calculate the transport coefficients in the aforementioned cases.
The other side of the problem is, of course, the computational power that is required to simulate through DNS such a physical situations with real astrophysical values that are often out of the range of the applicability of the test field method on human useful timescales.

When simulating an MHD environment in which a large scale magnetic field is generated the problem on the identification of the mechanism driving such a field arises naturally. For example [Hughes and Proctor, 2009] measured the substantial absence of $\alpha$ effect in simulations of convection thus finding a result that appears to be in conflict with others, like [Schrinner et al., 2005]. This problem is discussed in Paper II. In this work different ways to measure $\alpha$ are used and compared. In particular the imposed field method and the test field method are applied to an helically turbulent environment.

### 2.4 Magnetic Helicity

Magnetic helicity is defined as

$$H = \int A \cdot B \, dV, \tag{2.7}$$
that is the volume integral over a closed or periodic volume $V$ of the dot product of the vector potential $A$ and the relative magnetic field $B$.

The important role of magnetic helicity in plasma physics [Taylor, 1974, Jensen and Chu, 1984, Berger and Field, 1984], solar physics [Low, 1996, Rust and Kumar, 1994, Rust and Kumar, 1996], cosmology [Brandenburg et al., 1996, Field and Carroll, 2000, Christensson et al., 2005], and dynamo theory [Pouquet et al., 1976, Brandenburg and Subramanian, 2005] is due to the fact that it is a conserved quantity in ideal magnetohydrodynamics [Woltjer, 1958]. In the presence of finite magnetic diffusivity, the magnetic helicity can only change on a resistive time scale.

The conservation law of magnetic helicity is what allows to astrophysical bodies to have a magnetic fields the scale length of which is far larger than the size of the body itself, and then larger than those of the turbulent motions that are at the origin of the observed field. In presence of magnetic helicity flux, that is common in several astrophysical bodies like, for instance, the Sun, magnetic helicity is then not anymore a conserved quantity leading to a series of effects. One of the most important among these is the so-called alleviation of $\alpha$-quenching.

Magnetic helicity happens to have an interesting topological interpretation. When we deal with flux tubes it is indeed possible to write its value as the product of their linking number their fluxes [Brandenburg and Subramanian, 2005]. This means that a system of two or more interlocked flux tubes has a magnetic helicity, even if they are untwisted.

This way to look at magnetic helicity turns out to be important. When we are able to connect magnetic helicity to a physical structures we can then directly connect the stability, or the instability, of those to the value of $H$. In particular recent works (as paper III) showed as a structure of interlocked rings characterized by a finite value of $H$ can be more stable than a similar one in which $H = 0$. Indeed the conservation of magnetic helicity plays a major role both in the generation of dynamos and in the stability of structures.

Figure 2.3: The developing of small scale magnetic field shown in 3D visualisation. In the first panel $B_z$ is shown on the periphery of the computational domain, while $B_x$ is shown in the others (figure from paper II).
Aspects of the theory of galactic fields

The origin of the galactic magnetic fields could be explained in two ways. The first is the so-called primordial field theory: the field comes from the compression of an intergalactic field during galaxy formation. The second is the aforementioned dynamo theory: the field is the result of the amplification of a seed field through the action of a hydrodynamical dynamo. Dynamo theory is the most commonly accepted theory since a dynamic theory is required in order to explain the observations, as we have seen in chapter 1. We then use the theoretical approach and methods explained in chapter 2 and see how this applies to the study of galactic magnetic fields.

3.1 Dynamo generation of galactic fields

The first idea about the origin of the galactic magnetic field was that it had its origin prior to the formation of the galaxy, or at least the galactic disk. Later on Parker, in 1970, pointed out that dynamic motions would have expelled such a field on a timescale shorter than a billion of years. So the idea of dynamo generated fields came out: it would have been driven by cyclonic turbulence and differential rotation of the interstellar medium. During the formation of a galaxy there would have been the possibility of a Biermann battery that could lead to weak fields. Then this field has to be amplified by dynamo action, mainly powered by supernova-driven turbulence and stellar winds [Kulsrud, 1999]. We are talking about an environment in which turbulence is present since the beginning, so a cyclonic motion easily arises because of the Coriolis force due to the galactic rotation. In such a way any toroidal field, that is a field in the azimuthal direction in the galaxy, can be transformed into a poloidal field [Parker, 1955b]. The scale for this
Figure 3.1: A schematic representation of the $\alpha$-effect: a field line rises and twists creating an $\alpha$-like structure, thus generating a radial component from an azimuthal one (figure from [Parker, 1970]).

phenomenon to happen is that of the largest turbulent eddies. This effect is also known as $\alpha$ effect. On the other hand we have the so-called $\Omega$ effect: differential azimuthal rotation produces toroidal fields from poloidal ones.

The general assumption about the galactic field is that it satisfies the so-called “frozen-in” condition. This means that we can identify a magnetic line with a line of flux, that is the flux of particle in the interstellar medium corresponds to the configuration of the magnetic field. In terms of MHD equation we can write such a condition as

$$E = -u \times B,$$

in which $E$ is the electric field, $u$ the velocity field and $B$ the magnetic field in the medium we are considering. In general also the effect of diffusion has to be considered when we deal with magnetic diffusion and reconnection. In some situations in fact the frozen-in condition is not valid because of the presence of diffusion. This is, for example, the case of a stellar wind: its magnetic field satisfies the frozen-in condition but, when an obstacle as the magnetic field of a planet is encountered, the reconnection of the field lines cannot be avoided.
3.2 The generation of vorticity in the ISM

With the frozen-in condition holding, the $\alpha$ and $\Omega$ effects can be easily illustrated. In fact the field lines follow the flow-stream of matter, that is they can stretch, twist and raise giving birth to a tangled line from a straight one and so to a radial component from and azimuthal one and vice-versa like in Fig. 3.1. However it is important to point out that these effects can take place also in case in which the diffusion term is important, that is the case of low Reynolds number, even if in this case they are less efficient in producing a dynamo. In fact it has been verified both theoretically and through numerical simulations [Sur et al., 2008] that the transport coefficients $\alpha$ and $\eta$ are proportional to the magnetic Reynolds number.

Simulation of the interstellar turbulence have been conducted by [Gressel et al., 2008b] that calculated the $\alpha$-tensor as function of galactic height. They have also applied the test-field method to obtain a quantitative evaluation of the turbulent magnetic diffusivity in the context of fully dynamical MHD simulations of turbulence the ISM.

3.2 The generation of vorticity in the ISM

In order to go back to the dynamics in which magnetic fields were born in galaxies we summarize here a work that we have recently completed about some important concept regarding the development of vorticity in the ISM (see paper I). The generation of vorticity is schematically illustrated in Fig. 3.2. Expansion, compression as well as upward and downward motions are able to cause vorticity in the medium surrounding them. The ISM is the environment in which all the aforementioned magnetic fields develop. It is characterized by a rather complex dynamics due to several phenomena taking place in it. The most important among those phenomena are those
Aspects of the theory of galactic fields concerned with the explosions of stars, that is supernovae and superbubble: these events are able to influence a rather big part of a galaxy, with correlation length of 100 pc, and to drive turbulence in the ISM up to root mean square (hereafter rms) velocities of $\sim 10\ \text{km/s}$ \cite{Beck1996}. Simulations of these phenomena are nowadays able to reproduce pretty much the physics of these explosions: examples are the observed volume fractions of hot, warm, and cold gas \cite{Rosen1995,Korpi1999}, the statistics of pressure fluctuations \cite{MacLow2005}, the effects of the magnetic field \cite{deAvillez2005}, and even dynamo action \cite{Gressel2008a,Gissinger2009,Hanasz2009}.

Despite the fact that each supernova can roughly be described by radial expansion waves, in the aforementioned simulations it can be found a net production of vorticity. The development of vorticity can be seen as contradictory, since a spherical expansion should be irrotational in absence of any other forces, because the acceleration can be written as the gradient of a potential so leading to a zero vorticity

$$\omega = \nabla \times u = 0.$$  \hfill (3.2)

We will instead see that some vorticity is generated in such a condition either due to the fact that other effects occur to happen or some basic phenomena are added to simulate more realistic conditions of the ISM. In principle, vorticity could also be amplified by a dynamo effect for vorticity. Indeed in the equation describing its temporal evolution there is the $\nabla \times (u \times \omega)$ term, which is analogous to the induction term in dynamo theory for magnetic fields. In this case $\omega$ plays the role of the magnetic field. For the time being this effect has not been observed in simulations up to numerical resolution of $512^3$ meshpoints. \cite{Mee2006} showed that, when in isothermal conditions, only the viscous force can produce vorticity. This vorticity becomes negligible in the limit of large Reynolds numbers or small viscosity.

When adding magnetic fields the presence of vorticity cause effects that are illustrated in Fig. 3.3. Fields line are driven to change their topological global structure, they stretch and tangle up so driving $\alpha$ and $\Omega$ effects.

### 3.3 Alpha effect in galaxies

In analyzing the evolution of a magnetic field due to this phenomenon we take in account first the explosion of a single supernova and then the generation through superbubble. Supernovae are violent explosions of stars in which it could be radiated as much energy as in the whole solar life, that is up to $10^{52}$
Figure 3.3: A more complete view on the mechanism showed in fig. 3.2: the topology of the field changes while an expansion is occurring leading to different configurations upward and downward, that is towards the halo or midplane respectively. (Figure from [Brandenburg et al., 1990a]).
Aspects of the theory of galactic fields

Ergs in terms of kinetic and thermal energy. Superbubbles are giant cavity created by the action of powerful interstellar winds and supernova explosions in a cluster or association of early-type, that is very hot and luminous, stars. [Ferriere, 1992b] derives a simple analytical expression for $\alpha$ taking into account supernova and superbubbles explosions on the mean magnetic field. It is found that superbubbles are 7 times more efficient than isolated supernovae in the generation of a radial field from an azimuthal one, that is the $\alpha$-effect. In the vicinity of the Sun the vertical dependence of $\alpha$ can be well approximated, up to the basis of the halo, by the simple expression [Ferriere, 1992a]

$$\alpha(z) = 0.13 \text{ km/s} \frac{z}{100 \text{ pc}} \left[ 2e^{-|z|/200 \text{ pc}} - e^{-|z|/100 \text{ pc}} \right]. \quad (3.3)$$

From this expression we see, for example, how $\alpha$ is zero on the midplane. That is due to a symmetry property of the pseudoscalar $\alpha$. Moreover $\alpha$ increases with the galactic height $z$ up to the value of $0.13 \text{ km/s}$. The ratio $\alpha_z/\alpha_\phi$ is found to be negative in several works: [Ruediger and Kichatinov, 1993] find $\alpha_z/\alpha_\phi = -0.25$, [Ferriere, 1993b] found $\alpha_z/\alpha_\phi = -0.3$ using a different theoretical approach, [Brandenburg et al., 1990b] found $\alpha_z/\alpha_\phi = -3$ through MHD simulations. When all the explosions occurs around the midplane of a galaxy we can write

$$V_{\text{esc}} = \frac{z}{2\Delta \tau} \quad (3.4)$$

where $-V_{\text{esc}}$ indicate the $\alpha$-effect for the electromotive force along $y$, $\mathcal{E}_y = -V_{\text{esc}}B$, that is the antisymmetric part of the $\alpha$ tensor along $y$. The form of the $\alpha$ tensor in the expression relating the electromotive force to the mean magnetic field is [Ferriere, 1993b]

$$\alpha = \begin{pmatrix} \alpha_R & -V_{\text{esc}} & 0 \\ -V_{\text{esc}} & \alpha_\Phi & 0 \\ 0 & 0 & \alpha_z \end{pmatrix}$$

In general $\alpha$ is a pseudo tensor and the only way to obtain non vanishing diagonal components is to construct them using a combination of polar and axial vectors. This means that in absence of stratification and rotation these components would vanish. In Fig. 3.4 the components of the $\alpha$ tensor in function of galactic height are shown [Ferriere, 1993b].

For the turbulent magnetic diffusivity it has been obtained the following expression for the vertical and horizontal components [Ferri`ere, 2009]

$$\eta_v(z) = (6.1 \times 10^{24} \text{ cm}^2 \text{ s}^{-1}) \left( \frac{1}{1 + [z/62 \text{ pc}]^2} + \frac{[z/215 \text{ pc}]^{2.25}}{1 + [z/215 \text{ pc}]^{2.25}} \right) \quad (3.5)$$
3.4 Turbulent diffusion

The turbulent diffusion tensor $\eta_{ij}$ could attain finite values even in the case of complete homogeneity. In this case we would deal with an isotropic tensor, that is $\eta_{ij} = \eta_{\delta_{ij}}$. It is then important to study the dependence of the turbulent diffusivity tensor on the magnetic Reynolds number. We have in fact seen that in the mean field theory both $\alpha$ and $\eta$ are proportional to $R_m$, but they must stay finite even in the case of large magnetic Reynolds number. This is the subject of present studies mainly performed with direct numerical simulations (DNS), that is simulations in which no subgrid scale modeling is used and in which one requires to solve the real MHD equations instead of their approximations.

The first step is to consider subsonic flow since a direct simulations implies some limitations on the strength of the forcing. [Brandenburg and Del Sordo, 2010] use a Gaussian potential forcing $f(x,t) = \nabla \phi$, with

$$\phi = N \exp \left\{ -\left[ x - x_f(t) \right]^2 / R^2 \right\} ,$$

for the disk and the halo respectively. Vertical and horizontal magnetic diffusivity are shown in Fig. 3.5.

**Figure 3.4:** Left panel: components of the $\alpha$ tensor due to a vertical distribution of isolated Type II SNs. The velocity is normalized to the escape velocity $V_0$. We can note that $\alpha$ is vanishing on the midplane (figure from [Ferriere, 1993b]). Right panel: components of the $\alpha$ tensor due to isolated and clustered SNs together in the vicinity of the Sun. The dotted lines represent results of calculation while the solid ones are analytical fits (figure from [Ferriere, 1993b]).
Figure 3.5: Left panel: Vertical magnetic diffusivity due to isolated and clustered SNs together in the solar neighborhood as a function of Galactic height. The dotted lines represent results of calculation while the solid ones are analytical fits (figure from Ferrière, 2009). Right panel: Horizontal magnetic diffusivity due to isolated and clustered SNs together in the solar neighborhood as a function of Galactic height. The dotted lines represent results of calculation while the solid ones are analytical fits (figure from Ferrière, 2009).
where \( \mathbf{x} = (x, y, z) \) is the position vector, \( \mathbf{x}_f(t) \) is the random forcing position, \( R \) is the radius of the Gaussian and \( N \) is a normalization factor. We are then in the case of a potential flow simulated through a random force that is applied to the fluid in a Cartesian box. The first results that have been obtained are summarized in fig. (3.6). The turbulent diffusivity is normalized by \( \eta_0 \equiv \sqrt{\frac{u_{\text{rms}}}{3k_f}} \) and it is plotted in its dependence on \( R_m \). For low values of \( R_m \) the turbulent diffusivity seems to increase proportional to \( R_m^n \) with \( n \) between 1/2 and 1. For larger value of \( R_m \) \( \eta_t \) seems to level off at a value of about 20 times \( \eta_0 \) \[Brandenburg and Del Sordo, 2010\].

These results seem to suggest that the diffusion of magnetic field can be driven both by nearly irrotational and vortical turbulence approximatively with the same efficiency. However it’s clear how this is only a preliminary stage of the study. First of all the possible dependence on magnetic Prandtl number has to be investigated. Second, it’s important to clarify the dependence on magnetic field strength since we are dealing with a non linear regime. Moreover we need to study the case in which stratification and rotation are present, that is a physical case much closer to the real galactic one. In this situation in fact an \( \alpha \) effect could be driven as well as turbulent pumping leading to the study of a more complete situation. As we have seen, \[Ferriere, 1992a\] obtained some theoretical results for this model and then it would be spontaneous to try to verify those results through numerical simulations.
3.5 Where to go from here

We have seen how our research has been focused on different aspects of the problem of galactic magnetic fields. After having taken into account the general problem of the measurement of the $\alpha$ effect in numerical simulations (paper II), we have examined some aspects related to magnetic helicity (paper III). Both of them are general problems that, in our view, are necessary to address to go towards a better comprehension of the dynamo problem. The analysis of some aspect of turbulent diffusion in irrotational flows is a work in progress. This is a problem of general value that arises while studying spherical expansion and the production of turbulence: these are two important aspects of the dynamics of the ISM. After that, we have concentrated our efforts to the analysis of the generation of vorticity in the ISM, as shown in paper I, analyzing then a typical phenomena of the ISM, that is spherical explosions. This has been a purely dynamical study that has not touched any problem directly related to the production of magnetic fields. Nevertheless we know how turbulence and vorticity are two basic elements in dynamo theory, as for example shown by [Mee and Brandenburg, 2006]. Our simulations have shown that rotation is able to produce vorticity in a barotropic flow that is forced irrotationally. However the amount of vorticity is proportional to the Coriolis number, which is found to be small in galaxies. In the presence of gravity the system could become density-stratified. Gravity should not lead by itself to additional vorticity because the gravitational force is potential. However, it should lead vorticity and velocity aligned with each other, with a consequent production of helicity. That too should be proportional to the Coriolis number and should thus be small. This raises the question whether this is borne out by the analytical calculation of $\alpha_{ij}$ and $\eta_{ij}$ by [Ferriere, 1993a] and [Ferriere, 1993b]. In the analytic calculus, the effect of magnetic diffusivity was treated artificially by assuming that the flow renews suddenly. With the test-field method we can treat this now much more accurately. The next question is whether the effect is enhanced by adding baroclinicity. If so, $\alpha$ would be proportional to Mach number, because we found in paper I that the vorticity production is proportional to the Mach number. Our work has shown that vorticity production can easily be spurious. The question is therefore to what extent can helicity production also be spurious. By considering rotating stratified turbulence with potential forcing we should be able to address this question qualitatively.

We are then moving step by step, starting from general problems and going to specific ones and the final goal for the PhD project is a more complete study of the galactic dynamo. Our plan is to continue our studies in
3.6 My contribution to the papers

In Paper I I shared the work with my supervisor since the beginning. I have set up the different cases to be studied, I performed the analysis and wrote extensive parts of the paper.

Paper II was done at the very beginning of my PhD. I helped in running some of the simulations and took the occasion for learning much about the code. Then I also contribute in some parts of the text.

Paper III came out during some discussion in a course on solar physics. I had the idea to study the setup and the configuration described in the paper, then I run some of the simulations and wrote extensive part of the introduction and the results.
Bibliography


Vorticity production through rotation, shear, and baroclinicity

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\textbf{ABSTRACT}

\textbf{Context.} In the absence of rotation and shear, and under the assumption of constant temperature or specific entropy, purely potential forcing by localized expansion waves is known to produce irrotational flows that have no vorticity. To address the problem of vorticity generation in the interstellar medium in a systematic fashion. \textbf{Methods.} We use three-dimensional periodic box numerical simulations to investigate the various effects in isolation. \textbf{Results.} We find that for slow rotation, vorticity production in an isothermal gas is small in the sense that the ratio of the root-mean-square values of vorticity and velocity is small compared with the wavenumber of the energy carrying motions. Shear also raises the vorticity production, but no saturation is found. When the assumption of isothermality is dropped, there is significant vorticity production by the baroclinic term once the turbulence becomes supersonic. In galaxies, shear and rotation are estimated to be insufficient to produce significant amounts of vorticity, leaving therefore only the baroclinic term as the most favorable candidate. We also demonstrate vorticity production visually as a result of colliding shock fronts.

\textbf{Key words.} magnetohydrodynamics (MHD) – turbulence – Galaxies: magnetic fields – ISM: bubbles

1. Introduction

Turbulence in the interstellar medium (ISM) is believed to be driven by supernova explosions. Such events inject sufficient amounts of energy to sustain turbulence with rms velocities of \(\sim 10\,\text{km/s}\) and correlation lengths of up to \(100\,\text{pc}\) (Beck et al., 1996). Simulations of such events can be computationally quite demanding, because the bulk motions tend to be supersonic and the flows involve strong shocks in the vicinity of individual explosion sites, as was seen early on in two-dimensional simulations (Rosen & Bregman, 1995). Nevertheless, such simulations are able to reproduce a number of physical phenomena such as the observed volume fractions of hot, warm, and cold gas (Rosen et al., 1996; Korpi et al., 1999a), the statistics of pressure fluctuations (Mac Low et al., 2005), the effects of the magnetic field (de Avillez & Breitschwerdt, 2005), and even dynamo action (Gressel et al., 2008; Gissinger et al., 2009; Hanasz et al., 2009). These simulations tend to show the development of significant amounts of vorticity, which is at first glance surprising. Indeed, each supernova drives the gas radially outward and can roughly be described by radial expansion waves. In such a description, the turbulence is forced by a gradient of a potential that consists of a timedependent spherical blob at random locations. Obviously, such a forcing is irrotational, so no vorticity is produced.

Earlier work of Mee & Brandenburg (2006) showed that under isothermal conditions only the viscous force can produce vorticity and that this becomes negligible in the limit of large Reynolds numbers or small viscosity. In principle, vorticity can also be amplified akin to the dynamo effect by the \(\nabla \times (u \times \omega)\) term, which is analogous to the induction term in dynamo theory, where \(\omega\) plays the role of the magnetic field. However, neither this nor the viscosity effect were found to operate – even at numerical resolutions of up to \(512^3\) meshpoints. This disagreed with subsequent simulations of Federrath et al. (2010), who solved the isothermal inviscid Euler equations with irrotational forcing using the \textsc{Flash} Code. They found significant vorticity generation in proximity to shocks where some kind of effective numerical viscosity must have acted.

Given that under isothermal conditions, only viscosity can lead to vorticity production, one must ask whether numerical viscosity or effective viscosity needed to stabilize numerical codes might have contributed to the production of vorticity in some of the earlier works. Indeed, it is possible that the directional operator splitting used in the \textsc{Flash} Code may have been responsible for spurious vorticity generation in the work of Federrath et al. (2010); (R. Rosner, private communication). On the other hand, when cooling and heating functions are included to perform more realistic simulations of the ISM, vorticity could be produced by the baroclinic term. Furthermore, even in the isothermal case, in which the baroclinic term vanishes, vorticity could be produced if there is rotation and/or shear.

The baroclinic term results from taking the curl of the pressure gradient term and is proportional to the cross product of the gradients of pressure and density. This term can play an important role when the assumptions of isothermality or adiabaticity are relaxed. Indeed, the baroclinic term can also be written as the cross product of the gradients of entropy and temperature. This formulation highlights the need for non-ideal effects, because in the absence of any other heating or cooling mechanisms, the entropy is just driven by viscosity. Again, it is not obvious that in
the absence of additional heating and cooling much vorticity can be produced. On the other hand, it is clear that viscous heating must be significant even in the limit of vanishing viscosity, because the velocity gradients can be very large, especially in shocks. Of course, the assumption about additional heating and cooling is not realistic for the interstellar medium and will need to be relaxed. Finally, there are the effects of rotation and shear, that can contribute to the production of vorticity even in the absence of baroclinicity.

The goal of this paper is to study the relative importance of the individual effects that contribute to vorticity production. It is then advantageous to restrict oneself to simplifying conditions that allow one to identify the governing effects. An important simplification is the restriction to weakly supersonic conditions so that shocks and other sharp structures can still be resolved with just a uniform and constant viscosity. We also neglect the effects of stratification which can only indirectly contribute to vorticity production. In fact, a constant gravitational acceleration drops out when taking the curl. Only in the non-isothermal and non-isentropic case can gravity contribute to vorticity production by enhancing the effect of the baroclinic term. We begin with a preliminary discussion and a qualitative analysis of the important terms in the vorticity equation.

2. Preliminary considerations

We recall that in the absence of baroclinicity, rotation, and shear, the curl of the evolution equation of the velocity is given by (see, e.g., Mee & Brandenburg, 2006)

\[
\frac{\partial \omega}{\partial t} = \nabla \times (u \times \omega - \nabla \times \omega) + \nabla \times G,
\]

(1)

where \( \nu \) is the kinematic viscosity (assumed constant) and \( G \) is a part of the viscous force that has non-vanishing curl even when the flow is purely irrotational.

Here,

\[
S_{ij} = \frac{1}{2}(u_{ij} + u_{ji}) - \frac{1}{3} \delta_{ij} u_k\kappa
\]

(2)

is the traceless rate of strain matrix, and commas denote partial differentiation. The \( G \) term breaks the formal analogy with the induction equation. It is convenient to express the resulting rms vorticity in terms of the typical wavenumber \( k_\omega \) of vortical structures which we define as

\[
k_\omega = \omega_{\text{rms}}/u_{\text{rms}}.
\]

(3)

We monitor the ratio \( k_\omega/k_\ell \), where \( k_\ell \) is the adopted nominal forcing wavenumber.

In Mee & Brandenburg (2006), the resulting vorticity, expressed in terms of the ratio \( k_\omega/k_\ell \), was found to be zero within error bars. This result is compatible with the idea that the \( \nu \nabla \times G \) term in Equation (1) is insignificant for vorticity production. By contrast, in vortical turbulence and at moderate values of the Reynolds number, \( k_\omega/k_\ell \) is found to be of the order of unity (Brandenburg, 2001), although one should expect a mild increase proportional to the square root of the Reynolds number as this number increases.

2.1. Rotation

Rotation leads to the addition of the Coriolis force, \( 2\Omega \times u \), in the evolution equation for the velocity. Taking the curl, we obtain the vorticity equation (1) with two additional terms, both proportional to \( \Omega \), so we have

\[
\frac{\partial \omega}{\partial t} = \ldots - 2\Omega \nabla \times (u \times \omega) + 2\Omega \cdot \nabla u \times \omega,
\]

(4)

where the dots denote the other terms that we discussed already. In order to estimate the production of vorticity, one could derive an evolution equation for the enstrophy density, \( \frac{1}{2} \omega^2 \), by multiplying the right-hand side of Equations (1) and (4) by \( \omega \), and use a closure assumption for the resulting triple correlations. However, it is then difficult to obtain a useful prediction for \( \omega_{\text{rms}} \), because the right-hand side of such an equation would necessarily be proportional to \( \omega \) and would therefore vanish, unless \( \omega_{\text{rms}} \) was different from zero to begin with. Instead, we estimate \( \omega_{\text{rms}} \) by computing the rms value of \( \partial \omega/\partial t \) and replacing it by \( \omega_{\text{rms}}/\tau_\Omega \), where \( \tau_\Omega \) is a typical time scale of the problem. This leads to

\[
\omega_{\text{rms}} \approx 2\Omega \tau_\Omega \left( (\nabla \times u)_{\nabla}^2 + (\nabla u)_{\parallel}^2 \right)^{1/2},
\]

(5)

where \( \nabla \times \text{ and } \nabla \parallel \) denote derivatives in the directions perpendicular and parallel to the rotation axis and \( \nabla \parallel \) is the velocity vector perpendicular to the rotation axis. Using Cartesian coordinates where \( \Omega \) points in the \( z \) direction, we have

\[
\omega_{\text{rms}} \approx 2\Omega \tau_\Omega \left( u_{x,z}^2 + u_{y,z}^2 + u_{x}\kappa z + u_y^2 \right)^{1/2}.
\]

(6)

We expect \( \tau_\Omega \) to be comparable to the turnover time, \( \tau = (u_{\text{rms}}k_\ell)^{-1} \). We expect the rms values of the velocity derivative term in Equation (6) to be comparable to the rms velocity and some inverse length scale. Typically, one would expect it to be proportional to \( u_{\text{rms}}k_\ell \), although, again, there can be an additional dependence on the square root of the Reynolds number. However, for fixed Reynolds number, and not too rapid rotation, we expect \( \omega_{\text{rms}} \) to increase linearly with the Coriolis number, i.e.,

\[
\text{Co} = 2\Omega \tau, \quad \text{where } \tau = (u_{\text{rms}}k_\ell)^{-1}.
\]

(7)

Thus, we expect \( k_\omega/k_\ell = \text{St}_{\Omega} \text{Co} \), where we have defined an effective rotational Strouhal number,

\[
\text{St}_{\Omega} = \frac{\tau_{\text{eff}}}{\tau_{\Omega}} u_{\text{rms}}k_\ell.
\]

(8)

We regard this as a fit parameter that will emerge as a result of the simulations. We have here introduced the quantity \( \tau_{\text{eff}}/\tau_{\Omega} \), where \( \tau_{\text{eff}}/\tau_{\Omega} \) is given by the ratio of the velocity gradient terms divided by \( u_{\text{rms}}k_\ell \). However, for larger values of Co there may be a departure from a linear dependence between \( k_\omega/k_\ell \) and Co.

(We note that, apart from a possible \( 4\pi \) factor, the Coriolis number is just the inverse Rossby number.)

One aim of this paper is therefore to verify this dependence from simulations and to determine empirically the value of \( \tau_{\Omega} \).

2.2. Shear

In the presence of linear shear with \( u^S = (0, Sx, 0) \), the evolution equation for the departure from the mean shear
attains additional terms, \(-u^5 \nabla \cdot u - u \cdot \nabla u^5\). This implies a dependence of \(\omega_{\text{rms}}\) on \(S\), analogous to the \(Q\) dependence discussed above. In components form, this means that

\[
\omega_{\text{rms}} \approx S \tau_S \left( (u_{x,x} + u_{y,y})^2 + u_{x,z}^2 + u_{z,z}^2 + O(xu^5) \right)^{1/2},
\]

which is quite similar to Equation (5), except that in the penultimate term in angular brackets the indices are now interchanged, i.e. we now have \(u_{x,y}\) instead of \(u_{y,z}\). Again, \(\tau_S\) is a typical time scale of the problem and we expect it to be related to the turnover time \(\tau\). The \(O(xu^5)\) denotes the presence of additional terms that are proportional to \(x\) and to second derivatives of \(u\). However, when adopting the shearing box approximation with shearing-periodic boundaries (Goldreich & Lynden-Bell, 1965; Wisdom & Tremaine, 1988), each point in the \(xy\) plane is statistically equivalent. We would therefore not expect there to be a systematic \(x\) dependence, which corresponds to the assumption of Galilean invariance that is sometimes used in the study of turbulent transport coefficients in linear shear flows (Sridhar & Subramanian, 2009). We will postpone the possibility of additional terms until later. Analogously to the case with rotation, we expect \(\tau_S\) to be comparable to \(\tau = (\omega_{\text{rms}} k_T)^{-1}\), so we expect \(\omega_{\text{rms}}\) to be proportional to the shear parameter,

\[
\mathcal{S}_h = \mathcal{S} \equiv S/\omega_{\text{rms}} k_T,
\]

although for large values of \(|\mathcal{S}_h|\) we may expect departures from a linear dependence. Determining this dependence is another aim of this paper. Again, a linear dependence is characterized by the values of \(\tau_S\) and \(\tau_{\text{eff}}\), where, in analogy with the previous case with rotation, the ratio \(\tau_{\text{eff}}/\tau_S\) is given by the derivative term in Equation (9), normalized by \(\omega_{\text{rms}} k_T\). A convenient non-dimensional measure of the value of \(\tau_{\text{eff}}\) is what we call the shear Strouhal number,

\[
\mathcal{S}_T = \tau_{\text{eff}} \omega_{\text{rms}} k_T,
\]

which can be determined provided there is a range in \(\mathcal{S}_h\) over which \(\omega_{\text{rms}}\) increases linearly with \(\mathcal{S}_h\).

The study of vorticity production by rotation and shear is quite independent of thermodynamics and can in principle be studied even in the incompressible case. However, in the present paper we study this effect in the weakly compressible case of low Mach numbers and under the assumption of an isothermal equation of state, where the baroclinic term vanishes.

### 2.3. Baroclinicity

As mentioned in the introduction, the baroclinic term, proportional to \(\nabla \times \nabla p\), emerges when taking the curl of the pressure gradient term, \(\rho^{-1} \nabla p\). This term can also be written as

\[
\rho^{-1} \nabla p = \nabla h - T \nabla s,
\]

where \(h\) and \(s\) are specific enthalpy and specific entropy, respectively, and \(T\) is the temperature. Thus, we have

\[
\frac{\partial \omega}{\partial t} = \ldots + \nabla T \times \nabla s.
\]

In order to study the effect of the baroclinic term, it is useful to look at the dependence of the mean angle \(\theta\) between the gradients of \(s\) and \(T\), defined via

\[
\sin^2 \theta = \frac{\langle (\nabla T \times \nabla s)^2 \rangle}{\langle (\nabla T)^2 \rangle \langle (\nabla s)^2 \rangle}.
\]

An important aspect is then to study first the dependence of the rms values of the gradients of \(s\) and \(T\). We can do this by looking at a one-dimensional model where, of course, \(\theta = 0\).

Next, we need to determine \(\theta\) from three-dimensional simulations. The hope is then that we can express baroclinic vorticity production in the form

\[
k_{\omega}/k_T = 2 \mathcal{S}_\text{baro} \langle \nabla T \rangle_{\text{rms}} \langle \nabla s \rangle_{\text{rms}} \sin \theta / \omega_{\text{rms}} k_T^2,
\]

On dimensional grounds we expect the product of \((\nabla T)_{\text{rms}}\) and \((\nabla s)_{\text{rms}}\) to be of the order of \(u_{\text{rms}}^2 k_T^2\), and so a possible ansatz would be

\[
k_{\omega}/k_T = 2 \mathcal{S}_\text{baro} \sin \theta,
\]

where we have subsumed the scalings of \((\nabla T)_{\text{rms}}\) and \((\nabla s)_{\text{rms}}\) in that of an effective baroclinic Strouhal number \(\mathcal{S}_\text{baro}\).

An important issue is the fact that viscous heating leads to a continuous increases of the temperature. As a result, the sound speed changes and it becomes then impossible to study the behavior of the system in a steady state. In order to avoid this inconvenience, we add a volume cooling term that is non-vanishing when the local sound speed \(c_s\) is different from a given target value, \(c_{s0}\). Thus, in the presence of finite thermal diffusivity \(\chi\), and with a cooling term governed by a cooling time \(\tau_{\text{cool}}\), our entropy equation takes the form

\[
\frac{D s}{D t} = 2 \nu S^2 + \rho^{-1} \nabla \cdot (p \rho \chi \nabla T) - \frac{1}{\tau_{\text{cool}}} (c_s^2 - c_{s0}^2),
\]

where \(c_s\) is the adiabatic sound speed. We assume a perfect gas so that \(c_s^2 = (\gamma - 1) c_p T\), where \(\gamma = c_p/c_v = 5/3\) for a monatomic gas, and \(c_p\) and \(c_v\) are the specific heats at constant pressure and constant volume, respectively. The value of \(\tau_{\text{cool}}\) can have an influence on the results, so we need to consider different values. We express \(\tau_{\text{cool}}\) in terms of \(c_{s0}\) and \(k_T\), and define the nondimensional quantity \(\mathcal{S}_\text{cool} = \tau_{\text{cool}} c_{s0} k_T\).

### 3. The model

In this paper we solve the continuity equation for the density \(\rho\),

\[
\frac{D \rho}{D t} = - \nabla \cdot u,
\]

together with the momentum equation for the velocity \(u\),

\[
\frac{Du}{Dt} = - \rho^{-1} \nabla p - 2 \Omega \times u - Su_x \hat{y} + \nabla \phi + F_{\text{visc}},
\]

where \(D/Dt = \partial/\partial t + (u + u^c) \cdot \nabla\) is the advection operator with respect to the sum of turbulent flow \(u\) and laminar shear flow \(u^c\), \(p\) is the pressure, \(\phi\) is the forcing potential, and

\[
F_{\text{visc}} = \rho^{-1} \nabla \cdot (2 \nu S)
\]

is the viscous force, where \(S\) was defined in Equation (2). The forcing potential is given by

\[
\phi(x, t) = \phi_0 \exp \left\{ \frac{(x - x(t))^2}{R^2} \right\},
\]

where \(x = (x, y, z)\) is the position vector, \(x(t)\) is the random forcing position, \(R\) is the radius of the Gaussian, and
$N$ is a normalization factor. We consider two forms for the time dependence of $x_t$. First, we take $x_t$ such that the forcing is $\delta t$-correlated in time. Second, we include a forcing time $\delta t_{\text{force}}$ that defines the interval during which $x_t$ remains constant, after which the forcing changes again abruptly. In order that the prefactor $\phi$ has the same dimension as $\phi$, which is that of velocity squared, we choose $N = c_{\infty} \sqrt{c_{\infty} R/ \Delta t}$, where

$$\Delta t = \max (\delta t, \delta t_{\text{force}})$$

is the length of the time step, $\delta t$, in the $\delta t$-correlated case or equal to the mean interval $\delta t_{\text{force}}$ during which the force remains unchanged, depending on which is longer.

The work of Mee & Brandenburg (2006) showed that the peak of the energy spectrum depends on the radius $R$ of the Gaussian. Indeed, the Fourier transform of $\exp(-r^2/R^2)$ is also a Gaussian with $\exp(-k^2/k^2_1)$, where

$$k_1 = 2/R.$$ (23)

In the following we use this as our definition of $k_1$ and check a posteriori that this is close to the position of the peak of the energy spectrum. In the following, we characterize our simulations in terms of the ratio $k_1/k_1$, and consider values between 2 and 10.

We use the Pencil Code,\(^1\) which is a non-conservative, high-order, finite-difference code (sixth order in space and third order in time) for solving the compressible hydrodynamic and hydromagnetic equations. We adopt non-dimensional variables by measuring speed in units of a reference sound speed, $c_\infty$, and length in units of $1/k_1$, where $k_1$ is the smallest wavenumber in the periodic domain. This implies that the non-dimensional size of the domain is $(2\pi)^3$.

In order to study the effects of rotation and shear, we ignore entropy effects and restrict ourselves to an isothermal equation of state with constant sound speed $c_\infty$. This means that $\rho^{-1} \nabla p$ reduces to $c_\infty^2 \nabla \ln \rho = \nabla h$, which has vanishing curl. Here, $h = c_\infty^2 \ln \rho$ is the relevant enthalpy in the isothermal case. On the other hand, in order to study the effects of baroclinicity, we do need to allow the entropy to vary, so we also need to solve Equation (17), and study the dependence of $k_\omega/k_1$ on the Mach number,

$$Ma = u_{\text{rms}}/c_\infty.$$ (24)

In order to characterize the degree of turbulence, we define the Reynolds number based on the energy-carrying scale, corresponding to the typical wavenumber where the spectrum peaks, i.e.

$$Re = u_{\text{rms}}/\nu k_1.$$ (25)

For vortical turbulence, this definition is known to be a good measure of the ratio of the resulting turbulent viscosity divided by the molecular diffusivity (Yousef et al., 2003). The two numbers, $Ma$ and $Re$, can be varied by changing $\nu$

and/or the strength of the forcing. In all cases we use $\chi = \nu$. Another input parameter is the forcing Strouhal number

$$St_{\text{force}} = \tau_{\text{force}} u_{\text{rms}} k_1.$$ (26)

\(^1\)http://pencil-code.googlecode.com/
For Co is a good approximation for Co □ 03, i.e. St □ 0. Hence, St 0 □ Co, thus St □ Co ≈ 0.5, so the root of the sum of their squares is □ 0.5, St □ 0.58.

\[
\begin{array}{c|c|c|c|c}
\text{Co} & 0.11 & 0.35 & 0.99 & 2.80 \\
\hline
\langle \nabla \cdot \mathbf{u} \rangle_{\text{rms}} / \langle u_{\text{rms}} k_f \rangle & 1.24 & 1.20 & 1.21 & 1.04 \\

\text{rms} / \text{rms} & 0.76 & 0.74 & 0.75 & 0.70 \\

\text{rms} & 0.49 & 0.47 & 0.48 & 0.63 \\

\text{rms} & 0.49 & 0.47 & 0.48 & 0.63 \\

\text{rms} & 0.49 & 0.47 & 0.46 & 0.58 \\

\text{rms} & 0.32 & 0.33 & 0.35 & 0.42 \\

\text{rms} & 0.34 & 0.34 & 0.35 & 0.41 \\

\text{rms} & 0.35 & 0.34 & 0.29 & 0.18
\end{array}
\]
It turns out that in the presence of shear, some level of helicity production can never be avoided—even in the limit of small $Sh$. Again, this appears spurious and demonstrates the general sensitivity of vorticity generation on resolution effects. An additional problem is of course the finite size of the shearing box (Regev & Umurhan, 2008; Bodo et al., 2008), which may be responsible for spurious vorticity generation. On the other hand, there is vorticity generation even for large scale-separation ratios, $k_l/k_1 = 10$; see the dash-dotted line in Figure 5. This suggests the possibility of a more general problem that would not go away even in the limit of small eddies and small values of $|Sh|$. Nevertheless, there is a clear rise of $k_0/k_1$ when $|Sh| > 0.1$, which is in agreement with our expectations outlined in Section 2.2. However, the slope in this relationship is rather steep, $St_S \approx 6$. The velocity derivative terms are only slightly larger than in the case with rotation, corresponding to $\tau^S_{eff}/\tau_S \approx 1.5$; see also Table 2. Tentatively, this suggests that for comparable values of $Co$ and $Sh$, $\tau_S \gg \tau_0$. On the other hand, given that even for small values of $Sh$ there is spurious vorticity generation, we cannot be certain that the results are reliable for larger ones either. The case with shear must therefore remain somewhat inconclusive.

Finally, we consider the possibility of vorticity generation by the baroclinic term. In a preparatory step we study first the dependence of the product $(\nabla T)_{rms}(\nabla s)_{rms}$ on both $Ma$ and $Re$ in a one-dimensional model. In all cases we vary the strength of the forcing amplitude in the range $1 \leq \phi_0/c_s^2 \leq 500$ for different values of viscosity and cooling time. As we increase the value of $\phi_0$, the Reynolds number increases for a given value of the viscosity. For small values of $\phi_0$, the Mach number also increases linearly, where the ratio of $Ma/Re$ increases with increasing viscosity. However, for larger values of $Ma$ there is saturation and $Ma$ no longer increase with $\phi_0$.

Furthermore, in the range where $Ma$ still increases linearly with $\phi_0$, the rms value of the entropy gradient also increases, but it also saturates when $Ma$ saturates. The rms value of the temperature gradient, however, decreases with

Table 2. Similar to Table 1, but for the case with shear.

<table>
<thead>
<tr>
<th>$Sh$</th>
<th>$-0.01$</th>
<th>$-0.03$</th>
<th>$-0.06$</th>
<th>$-0.12$</th>
<th>$-0.26$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\nabla u \cdot u)<em>{rms}/u</em>{rms}k_l$</td>
<td>1.35</td>
<td>1.36</td>
<td>1.36</td>
<td>1.37</td>
<td>0.87</td>
</tr>
<tr>
<td>$u_{x,rms}/u_{rms}k_l$</td>
<td>0.87</td>
<td>0.87</td>
<td>0.90</td>
<td>0.97</td>
<td>0.75</td>
</tr>
<tr>
<td>$u_{x,rms}/u_{rms}k_l$</td>
<td>0.81</td>
<td>0.81</td>
<td>0.76</td>
<td>0.66</td>
<td>0.47</td>
</tr>
<tr>
<td>$u_{y,rms}/u_{rms}k_l$</td>
<td>0.51</td>
<td>0.50</td>
<td>0.51</td>
<td>0.53</td>
<td>0.72</td>
</tr>
<tr>
<td>$u_{z,rms}/u_{rms}k_l$</td>
<td>0.46</td>
<td>0.47</td>
<td>0.47</td>
<td>0.46</td>
<td>0.74</td>
</tr>
<tr>
<td>$\angle u_{x}/u_{rms}$</td>
<td>0.46</td>
<td>0.48</td>
<td>0.46</td>
<td>0.43</td>
<td>0.57</td>
</tr>
<tr>
<td>$\angle u_{y}/u_{rms}$</td>
<td>0.37</td>
<td>0.38</td>
<td>0.37</td>
<td>0.36</td>
<td>0.25</td>
</tr>
<tr>
<td>$\angle u_{z}/u_{rms}$</td>
<td>0.31</td>
<td>0.31</td>
<td>0.32</td>
<td>0.34</td>
<td>0.56</td>
</tr>
<tr>
<td>$\angle u_{z}/u_{rms}$</td>
<td>0.32</td>
<td>0.31</td>
<td>0.31</td>
<td>0.31</td>
<td>0.25</td>
</tr>
</tbody>
</table>
increasing values of $\phi_0$, but this seems to be a special property of the one-dimensional model that is not borne out by the three-dimensional simulations where it stays approximately constant.

Remarkably, the results are fairly independent of the cooling time, except that the break point where $(\nabla s)_{\text{rms}}$ saturates occurs for smaller values of $\phi_0$ as we increase the cooling time; see Figure 6. This break point is also related to the point where the Mach number saturates, as can be seen from Figure 7.

However, for longer cooling times there can be longer transients, making it difficult to obtain good averages. Therefore we focus in the rest of this paper on the case of shorter cooling times using $S_{\text{cool}} = 0.2$. Another remarkable result is that the normalized value of $(\nabla T)_{\text{rms}}(\nabla s)_{\text{rms}}$ is always of the order of about $10^{-3}$, independent of resolution, cooling time, or the value of the viscosity.

Most of the basic features of the one-dimensional model are also reproduced by two- and three-dimensional calculations. Two-dimensional simulations have the advantage of being easily visualized and are therefore best suited for illustrating vorticity production by the baroclinic term. In Figure 8 we demonstrate that vorticity production is associated with the interaction between the fronts of different expansion waves. In this example we chose $\delta t_{\text{force}}c_0/R = 0.1$, so the first expansion wave is launched at $t = 0$ and the second one at $t = 0.1$. The top row of Figure 8 shows that at $t = 0.09$, i.e. just before launching the second expansion wave, the baroclinic term and the vorticity are still just at the noise level of the calculation. Even under our weakly supersonic conditions shock surfaces are well localized and the zones of maximum production of vorticity appear to be those in which the fronts encounter each other. Here we have used $\phi_0/c_0 = 100$, $\nu = \chi = 0.1c_0R$, with 512$^2$ mesh points. Only the inner part of the domain is shown.

In order to have a more accurate quantitative determination of vorticity production, we now consider three-
Fig. 9. Dependence of Ma and Re, as well as the rms values of temperature and entropy on \( \phi_0 \) for \( \nu/c_s R = 1 \).

dimensional models. In Figure 9 we show the dependence of various quantities on \( \phi_0 \) for \( St_{\text{cool}} = 0.2 \) and \( \nu/c_s R = 1 \).

In all cases we use \( 128^3 \) mesh points and average the results over between 20 and 70 turnover times.

Note that here Re \( \approx 0.05 \phi_0/c_s^2 \). Given that Re depends inverse proportionally on \( \nu/Re_{\phi_0} \), we can also write Re \( \approx 0.05 \phi_0 R/c_s \). The Mach number saturates at about \( Ma = 3 \), and the rms value of the entropy gradient increases up until this point. Given that the rms value of the temperature gradient also stays approximately constant, we find a weak increase of \( \langle \nabla T \rangle_{\text{rms}} \langle \nabla s \rangle_{\text{rms}} \). The value of \( \langle \nabla T \times \nabla s \rangle_{\text{rms}} \) is always found to be a certain fraction below this value, resulting in a typical baroclinic angle of about 45 degrees; see the third panel of Figure 9. Finally, the amount of vorticity production in terms of \( k_{\omega}/k_t \) is about 0.3 for \( \phi_0/c_s^2 \approx 20 \). For smaller values, on the other hand, there is an approximately linear increase with \( k_{\omega}/k_t \approx 0.014 \phi_0/c_s^2 \).

The possibility of spurious vorticity is easily eliminated in this case by looking at enstrophy spectra; see Figure 10, where we compare \( E_{\omega}(k) \) with \( k^2 E_K(k) \). All spectra fall off rapidly with increasing \( k \). Thus, even though the initial vorticity generation occurred evidently at the smallest available scales, once the flow becomes fully developed, most of the enstrophy resides at scales equal to or larger than the driving scale. Furthermore, the spectra of \( E_{\omega}(k) \) and \( k^2 E_K(k) \) are close together, illustrating that the vorticity is close to its maximal value.

5. Applications

The level of vorticity that is produced in the usual case of solenoidal forcing of the turbulence is such that \( k_{\omega}/k_t \approx 1 \) (see, e.g., Brandenburg, 2001). For turbulence whose forcing has finite correlation time (\( St_{\text{force}} = 0.3 \), for example), and small values of Re, we have \( k_{\omega}/k_t = O(1) \) when \( Co > 10 \). However, for larger values of Re, the turbulence becomes vortical already for smaller values of Co. Comparing with the galaxy, we have \( \Omega \approx 10^{-15} \text{s}^{-1} \), \( u_{\text{rms}} = 10 \text{km/s} \), and an estimated correlation length of about 70 pc, so \( k_l = 3 \times 10^{-5} \text{cm} \), so Co = 0.07, which is rather small. Thus, rotation may not be able to produce sufficient levels of vorticity. Given that in galaxies with flat rotation curves, \( S \approx -\Omega \), shear should not be very efficient either. However, the Mach numbers are undoubtedly larger than unity in the interstellar medium, so this should lead to values of \( k_{\omega}/k_t \approx 0.3 \), which is the saturation value found in Figure 9. Given that one of the reasons for studying the production of vorticity is the question of dynamo action, we should point out that such values of \( k_{\omega}/k_t \) are large enough for the small-scale dynamo. Large-scale dynamo action should be possible in galaxies as well, because of their large length scales, but it suffers from the well-known problem of a small growth rate. It then remains difficult to explain large-scale magnetic fields in very young galaxies (Beck et al., 1996).

The question of vorticity generation is also important in studies of the very early Universe, where phase transition bubbles are believed to be generated in connection with the electroweak phase transition (Kajantie & Kurki-Suonio, 1986; Ignatius et al., 1994). Here the equation of
state is that of a relativistic fluid, \( p = \rho c^2 / 3 \), where \( c \) is the speed of light. Thus, there is no baroclinic term and no obvious source of vorticity. However, the relativistic equation of state may be modified at small length scales, but it is not clear that this can facilitate significant vorticity production.

6. Conclusions

The present work has demonstrated that vorticity production is actually quite ubiquitous once there is rotation, shear, or baroclinicity. This implies that the assumption of potential flows as a model for interstellar turbulence might be of academic interest and could only be realized under special conditions of weak forcing, weak rotation, and no shear. In galaxies, however, the shear and Coriolis number are well below unity, leaving only the baroclinic term as a viable candidate for the production of vorticity. This agrees with early work of Korpi et al. (1999b), who analyzed the production terms in supersonic, supernova-driven turbulence quantitatively. We have also observed how vorticity is mainly produced close to shock front encounters. This motivates a more detailed investigation of these zones as the next step in the study of vorticity generation in the interstellar medium. It should also be pointed out that the baroclinic term corresponds to the battery term in the induction equation (Kulsrud et al. 1997; Brandenburg & Subramanian 2005). Thus, when studying the possibility of dynamo action, this battery term provides an intrinsic and well defined seed for the dynamo and should therefore be included as well.

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The α effect with imposed and dynamo-generated magnetic fields

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ABSTRACT

Estimates for the non-linear α effect in helical turbulence with an applied magnetic field are presented using two different approaches: the imposed-field method where the electromotive force owing to the applied field is used, and the test-field method where separate evolution equations are solved for a set of different test fields. Both approaches agree for stronger fields, but there are apparent discrepancies for weaker fields that can be explained by the influence of dynamo-generated magnetic fields on the scale of the domain that are referred to as meso-scale magnetic fields. Examples are discussed where these meso-scale fields can lead to both drastically overestimated and underestimated values of α compared with the kinematic case. It is demonstrated that the kinematic value can be recovered by resetting the fluctuating magnetic field to zero in regular time intervals. It is concluded that this is the preferred technique both for the imposed-field and the test-field methods.

Key words: hydrodynamics – magnetic fields – MHD – turbulence.

1 INTRODUCTION

The α effect is commonly used to describe the evolution of the large-scale magnetic field in hydromagnetic dynamos (Moffatt 1978; Parker 1979; Krause & Rädler 1980). However, the α effect is not the only known mechanism for explaining the generation of large-scale magnetic fields. Two more effects have been discussed in cases where there is shear in the system: the incoherent alpha-shear dynamo (Vishniac & Brandenburg 1997; Sokolov 1997; Silant’ev 2000; Proctor 2007) and the shear-current effect (Rogachevskii & Kleedorin 2003, 2004). In order to provide some understanding of the magnetic field generation in astrophysical bodies such as the Sun or the Galaxy, or at least in numerical simulations of these systems, it is of interest to be able to identify the underlying mechanism.

Astrophysical dynamos are usually confined to finite domains harbouring turbulent fluid motion. Both the Sun and the Galaxy are gravitationally stratified and rotating, which makes the turbulence non-mirror symmetric, thus leading to an α effect. In addition, the rotation is non-uniform, which leads to a strong amplification of the magnetic field in the toroidal direction, as well as other effects such as those mentioned above. Instead of simulating such systems with all their ingredients, it is useful to simplify the set-up by restricting oneself to Cartesian domains that can be thought to represent a part of the full domain. At low magnetic Reynolds numbers, i.e. when the effects of induction are comparable to those of magnetic diffusion, the α effect can clearly be identified in simulations of convection in Cartesian domains; see Brandenburg et al. (1990). Here, α has been determined by applying a uniform magnetic field across the simulation domain and measuring the resulting electromotive force. This method is referred to as the imposed-field method. However, in subsequent years simulations at larger magnetic Reynolds numbers have revealed problems in that the resulting α becomes smaller and strongly fluctuating in time. This was first found in simulations where the turbulence is caused by an externally imposed body force (Cattaneo & Hughes 1996; Hughes & Cattaneo 2008), but it was later also found for convection (Cattaneo & Hughes 2006). This suggested that the mean-field approach may be seriously flawed (Cattaneo & Hughes 2009).

Meanwhile, there have been a number of simulations of convection where large-scale magnetic fields are being generated. Such systems include simulations not only in spherical shells (Browning et al. 2006; Brown et al. 2007), but also in Cartesian domains (Käpylä, Korpi & Brandenburg 2008, 2009a; Hughes & Proctor 2009). However, the absence of a significant α effect in some of these simulations led Hughes & Proctor (2009) to the conclusion that such magnetic fields can only be explained by other mechanisms such as the incoherent alpha-shear dynamo or the shear-current effect. Such an explanation seems to be in conflict with earlier claims of a finite α effect as determined by the test-field method of Schrinner et al. (2005, 2007), and in particular with recent results for convection (Käpylä, Korpi & Brandenburg 2009b). The purpose of the present paper is therefore to discuss possible reasons for conflicting results that are based on different methods. The idea is to compare measurements of the α effect using both the imposed-field method and the test-field method. We consider here the case of helically forced turbulence in a triply periodic domain. This case is believed to be well understood. We expect
α to be catastrophically quenched, i.e. α is suppressed for field strengths exceeding the Zeldovich (1957) value of $R_m^{-1/2} B_0$, where $R_m$ is the equipartition field strength where kinetic and magnetic energy densities are comparable. The importance of the Zeldovich field strength was emphasized by Gruzinov & Diamond (1994) in connection with catastrophic quenching resulting from magnetic helicity conservation.

In this paper we focus on the case of moderate values of $R_m$ of around 30. This is small by comparison with astrophysical applications, but it is large compared with the critical value for dynamo action in fully helical turbulence (Brandenburg 2001), which occurs for $R_m \gtrsim 1$ in our definition of $R_m$ based on the wavenumber of the scale of the energy carrying eddies, i.e. the forcing wavenumber. In addition, we only consider cases with a magnetic Prandtl number of unity. However, this should not worry us too much, because we know that the large-scale dynamo works independently of the value of the magnetic Prandtl number (Mininni 2007; Brandenburg 2009).

2 HELICAL TURBULENCE AND α EFFECT

2.1 Forced turbulence simulations

Throughout this paper we consider hydromagnetic turbulence in the presence of a mean magnetic field $B_0$ using triply periodic boundary conditions. The total magnetic field is written as $B = B_0 + \nabla \times A$, where $A$ is the magnetic vector potential. We employ an isothermal equation of state where the pressure is proportional to the density, $p = \rho c_s^2$, with $c_s$ being the isothermal sound speed. The governing evolution equations for logarithmic density $\ln \rho$, velocity $U$, together with $A$, are given by

$$\frac{d \ln \rho}{dt} = -\nabla\cdot U, \quad \frac{dU}{dt} = J \times (B_0 + B) + f + F_{\text{visc}} - c_s^2 \nabla \ln \rho, \quad \frac{\partial A}{\partial t} = U \times (B_0 + B) + \eta \nabla^2 A, \quad \left( \begin{array}{c} \delta q \end{array} \right) = \left( \begin{array}{c} 0 \end{array} \right).$$

where $B_0 + B$ is the total magnetic field, but since $B_0 = \text{const}$ does not enter in the mean current density, which is given by $J = \nabla \times B / \mu_0$, where $\mu_0$ is the permeability. Furthermore, $d/dt = \partial / \partial t + U \cdot \nabla$ is the advective derivative, $F_{\text{visc}} = \rho^{-1} \nabla \cdot \mathbf{S}$ is the viscous force, $\mathbf{S} = (1/2)(U_{ij} + U_{ji}) - (1/3)\delta_{ij} \nabla \cdot U$ is the traceless rate of strain tensor and $f$ is a random forcing function consisting of plane transversal waves with random wavevectors $k$ such that $|k|$ lies in a band around a given forcing wavenumber $k_0$. The vector $k$ changes randomly from one time-step to the next. This method is described for example in Haugen, Brandenburg & Dobler (2004). The forcing amplitude is chosen so that the Mach number $Ma = u_{\text{rms}} / c_s$ is about 0.1.

We consider a domain of size $L_x \times L_y \times L_z$. We use $L_x = L_y = L_z = 2\pi/k_1$ in all cases. Our model is characterized by the choice of magnetic Reynolds and Prandtl numbers, defined here via

$$R_m = u_{\text{rms}} / \eta k_1, \quad P_m = v / \eta. \quad (4)$$

We start the simulations with zero initial magnetic field, so the field is entirely produced by the imposed field. The value of the magnetic field will be expressed in units of the equipartition value

$$B_{\text{eq}} = (\mu_0 \rho U^2)^{1/2}. \quad (5)$$

We consider values of $B_0 / B_{\text{eq}}$ from 0.06 to 20 along with a magnetic Reynolds number of about 26, adequate to support dynamo action.

2.2 α from the imposed-field method

The present simulations allow us to determine directly the α effect under the assumption that the relevant mean field is given by volume averages, denoted here by angular brackets. Given that the magnetic field is written as $\mathbf{B} = \nabla \times \mathbf{A}$ where $\mathbf{A}$ is also triply periodic, we have $\langle \mathbf{B} \rangle = \mathbf{0}$. We can determine the volume-averaged electromotive force:

$$\langle \mathbf{E} \rangle = \langle \mathbf{E}(t) \rangle \equiv (u \times \mathbf{b}), \quad (6)$$

where $u = \mathbf{U} - \langle \mathbf{U} \rangle$ and $\mathbf{b} = \mathbf{B}$ are the fluctuating components of velocity and magnetic field, and $\langle \mathbf{B} \rangle = (\nabla \times \mathbf{A}) = \mathbf{0}$. For mean fields defined as volume averages, and because of periodic boundary conditions, we have $\langle \mathbf{J} \rangle = \mathbf{0}$. Under isotropic conditions there is therefore only the α effect connecting $\langle \mathbf{E} \rangle$ with $\mathbf{B}_0$ via $\langle \mathbf{E} \rangle = \alpha_{\text{imp}} \mathbf{B}_0$, so

$$\alpha_{\text{imp}} = \langle \mathbf{E} \rangle \cdot \mathbf{B}_0 / B_0^2. \quad (7)$$

In all cases reported below we assume $B_0 = (B_0, 0, 0)$. Note that $\nabla \times \langle \mathbf{E} \rangle = \mathbf{0}$ and therefore our time-constant imposed field is self-consistent.

2.3 α from the test-field method

A favoured method of determining the full $\alpha_q$ tensor is by using the test-field method (Schrinner et al. 2005, 2007), where one solves, in addition to equations (1)–(3), a set of equations. In the special case of volume averages this set of equations simplifies to

$$\frac{\partial \mathbf{a}^q}{\partial t} = \mathbf{U} \times \mathbf{b}^q + u \times (\mathbf{B}_0 + \mathbf{B}^q) + u \times \mathbf{b}^q - \mathbf{U} \times \mathbf{b}^q + \eta \nabla^2 \mathbf{a}^q, \quad (8)$$

where $\mathbf{b}^q = \nabla \times \mathbf{a}^q$ with $q = 1$ or 2 denotes the response to each of the two test fields $B^q$. Throughout this paper, overbars denote planar averages. Later we consider arbitrary planar averages and denote their normals by superscripts, but here we restrict ourselves to xy averages. We use two different constant test fields:

$$\mathbf{B}^q = (B, 0, 0), \quad \mathbf{B}^q = (0, B, 0), \quad (9)$$

where $B = \text{const}$ is the magnitude of the test field, but its actual value is of no direct significance, because the $\mathbf{B}$ factor cancels in the calculation of $\alpha$.

However, given that the test-field equations are linear in $\mathbf{b}^q$, this field can grow exponentially due to dynamo action. When $|\mathbf{b}^q|$ becomes larger than about 20 times the value of $\mathbf{B}$, the determination of $\alpha$ becomes increasingly inaccurate, so it is advisable to reset $\mathbf{b}^q$ to zero in regular intervals (Sur, Brandenburg & Subramanian 2008). We calculate the corresponding values of the electromotive force $\langle \mathbf{E} \rangle^q = (u \times \mathbf{b}^q)$ to determine the components

$$\alpha_{\text{eq}} = \langle \mathbf{E} \rangle^q / B. \quad (10)$$

This corresponds to the special case $k = 0$ when considering sinusoidal and cosinusoidal test functions described elsewhere (Brandenburg, Rädler & Schrinner 2008a).

Even though the test-field equations themselves are linear, the flow field is affected by the actual magnetic field (which is different from the test field), so the resulting $\alpha$ tensor is being affected.

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2.4 α in the presence of meso-scale fields

The relevant mean field may not just be the imposed field with wavenumber \( k = 0 \), but it may well be a field with wavenumber \( k = k_1 \). Such a field would vanish under volume averaging, but it would still produce finite values of \( \langle \hat{B}_i \hat{B}_j \rangle \). For the diagonal components of \( \alpha_0 \) we can write

\[
\langle \alpha_{\alpha_0} \rangle = \alpha_1 + \epsilon_x \alpha_2, \quad \langle \alpha_{\beta_0} \rangle = \alpha_1 + \epsilon_y \alpha_2,
\]

where the factors

\[
\epsilon_x = \langle \hat{B}_z^2 \rangle \quad \text{and} \quad \epsilon_y = \langle \hat{B}_z^2 \rangle
\]

quantify the weight of the \( \alpha_2 \) term. For a purely uniform field pointing in the \( x \) direction we have \( \epsilon_x = 1 \) and \( \epsilon_y = 0 \), while for a Beltrami field of the form \( \hat{B} = \text{const} \sin k z, \text{sin} k z, 0 \) we have \( \epsilon_x = \epsilon_y = 1/2 \).

In practice we will have a mixture between the imposed field (below sometimes referred to as large-scale field) and a dynamo-generated magnetic field with typical wavenumber \( k = k_1 \) (below sometimes referred to as meso-scale magnetic field). The solution to the test-field equations, \( \hat{B}^\alpha \), can also develop meso-scale fields with wavevectors in the \( x \) or \( y \) directions, but not in the \( z \) direction, because that component is removed by the term \( \hat{B} \times \hat{b} \) in equation (8). Table 1 highlights the difference between imposed, meso-scale and test fields. We denote the ratio of the strengths of imposed and meso-scale fields as \( \beta = B_0 / B_1 \) and distinguish three (and later four) different cases, depending on the direction of the wavevector of the Beltrami field.

The first case is referred to as the X branch, because the wavevector of the Beltrami field points in the \( x \) direction. To calculate \( \epsilon_x \), there is, in addition to the imposed field \( B_0 \), a Beltrami field \( B_1(0, \cos k x, \sin k x) \), which does not have a component in the \( x \) direction. Thus, \( \beta = B_0 / B_1 \), and since \( B = (B_0, B_1 \cos k x, B_1 \sin k x) \), we have \( \beta^2 = B_0^2 + B_1^2 \), so \( \epsilon_x = \frac{B_0}{B_1} = B_0/B_1^2 + B_1^2 \), or \( \epsilon_x = \beta^2/(1 + \beta^2) \). Likewise, with \( B_1 = B_1 \cos k x \), we find for the volume average or, in this case, the \( x \) average \( \langle B_1^2 \rangle = B_1^2/2 \), so \( \epsilon_x = 1/2[1 + \beta^2] \).

The next case is referred to as the Y branch, because the wavevector of the Beltrami field points in the \( y \) direction. Thus, we have

Table 1. Overview of the different types of fields and their meaning.

<table>
<thead>
<tr>
<th>Field</th>
<th>Symbol</th>
<th>Magn</th>
<th>Induct. eqn</th>
<th>Test-field eqn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Imposed field</td>
<td>( B_0 )</td>
<td>( B_0 )</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Meso-scale field</td>
<td>( \hat{B} )</td>
<td>( B_1 )</td>
<td>Yes</td>
<td>–</td>
</tr>
<tr>
<td>Test field</td>
<td>( \hat{B}_1 )</td>
<td>( B )</td>
<td>–</td>
<td>Yes</td>
</tr>
<tr>
<td>Test field response</td>
<td>( b^\alpha )</td>
<td>–</td>
<td>–</td>
<td>Yes</td>
</tr>
</tbody>
</table>

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Figure 1. Plot of the integrals \( I_1(\beta) \) and \( I_2(\beta) \).

\[
B = (B_0 + B_1 \sin k y, 0, B_1 \cos k y), \quad \hat{B}^2 = B_0^2 + 2B_0 B_1 \sin k y + B_1^2 \quad (\text{this is no longer independent of position, so the volume average or, in this case, the } y \text{ average has to be obtained by integration}) \]

Thus, we write \( \epsilon_x = I_1(\beta) \) where we have defined

\[
I_1(\beta) = \frac{1}{2} \int_0^{2\pi} \frac{\beta^2 \sin^2 \theta}{\beta^2 + 2\beta \sin \theta + 1} d\theta = \begin{cases} \frac{1}{2} & \beta^2 \leq 1, \\ 1 & \beta^2 \geq 1. \end{cases}
\]

where \( \theta = k y \) has been introduced as dummy variable. Since \( B_1 = 0 \) in this case, we have \( \epsilon_y = 0 \).

Finally, for the Y branch, where the wavevector of the Beltrami field points in the \( z \) direction, we have \( \hat{B} = (B_0, B_1 \cos k z, B_1 \sin k z, 0) \), we find \( \epsilon_x = I_1(\beta) \) and \( \epsilon_y = I_2(\beta) \) with

\[
I_2(\beta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos^2 \theta}{\beta^2 + 2\beta \cos \theta + 1} d\theta = \begin{cases} \frac{1}{2(1 + \beta^2)} & \beta^2 < 1, \\ \frac{1}{2(1 - \beta^2)} & \beta^2 > 1. \end{cases}
\]

where \( I_2(\beta) = (1 + \beta^2)/[2(1 - \beta^2)] \) and \( \theta = k z \) has been used as a dummy variable. A graphical representation of the integrals is given in Fig. 1 and a summary of the expressions for \( \epsilon_x(\beta) \) and \( \epsilon_y(\beta) \) as well as \( \epsilon_x(0) \) and \( \epsilon_y(0) \) for the \( X \), \( Y \) and \( Z \) branches is given in Table 2.

Table 2. Summary of the expressions for \( \epsilon_x(\beta) \) and \( \epsilon_y(\beta) \) as well as \( \epsilon_x(0) \) and \( \epsilon_y(0) \) for the \( X \), \( Y \) and \( Z \) branches.

<table>
<thead>
<tr>
<th>Branch</th>
<th>( \epsilon_x(\beta) )</th>
<th>( \epsilon_y(\beta) )</th>
<th>( \epsilon_x(0) )</th>
<th>( \epsilon_y(0) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>( I_1(\beta) )</td>
<td>( 1/2(1 + \beta^2) )</td>
<td>0</td>
<td>1/2</td>
</tr>
<tr>
<td>Y</td>
<td>( I_1(\beta) )</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>Z</td>
<td>( I_1(\beta) )</td>
<td>( I_2(\beta) )</td>
<td>1/2</td>
<td>1/2</td>
</tr>
</tbody>
</table>

3 RESULTS

We have performed simulations for values of \( B_0 \) in the range \( 0.06 \leq R_m^{1/2} B_0 / B_m \leq 20 \) for \( R_m \approx 26 \) and \( P_m = 1 \). In all cases we use \( B_0/B_1 = 3 \), which is big enough to allow a meso-scale magnetic field of wavenumber \( k_1 \) to develop within the domain; see Fig. 2. We did not initially anticipate the importance of the meso-scale fields. Different runs were found to exhibit rather different behaviour which turned out to be related to their random positioning on different branches. We used the existing results from different branches as initial conditions for neighbouring values of \( B_0 \).

In this paper, error bars are estimated from the averages obtained from any of three equally long subsections of the full time series.
The error bars are comparable with the typical scatter of the data points, but they are not shown because they would make the figure harder to read. Note that the results in this section consider saturated fields. The opposite case will be considered in Section 4.

### 3.1 Different branches

The resulting values of \( \alpha \) are shown in Fig. 3. For strong imposed magnetic fields, \( R_m B_0^2/B_{eq}^2 > 1 \), the resulting dependence of \( \alpha \) on \( B_0 \) obeys the standard catastrophic quenching formula for the case of a uniform magnetic field (Vainshtein & Cattaneo 1992):

\[ \alpha_{\text{fit}} = \frac{\alpha_0}{1 + R_m B_0^2/B_{eq}^2} \quad \text{(for } B = B_0 = \text{const only)}, \]

where \( \alpha_0 = -(1/3) u_{rms} \) is the relevant kinematic reference value for fully helical turbulence with negative helicity and \( R_m > 1 \) (Sur et al. 2008). We treat \( R_m \) as an empirical fit parameter that is proportional to \( R_m \) and find that \( R_m \approx 0.4 \) gives a reasonably good fit; see the dash--dotted line in Fig. 3. The existence of such an empirical factor might be related to the fact that the relevant quantity could be the width of the magnetic inertial range, and that this is not precisely equal to \( R_m \). For \( R_m B_0^2/B_{eq}^2 > 1 \), a similar result is also reproduced using the test-field method, although \( \alpha_{\text{fit}} \) is typically somewhat larger than \( \alpha_{\text{imp}} \).

For weak imposed magnetic fields, \( R_m B_0^2/B_{eq}^2 < 1 \), apparent discrepancies are found between the imposed-field method and the test-field method. In fact, in the graphical representation in Fig. 3 the results can be subdivided into four different branches that we refer to as branches X, Y, Z, and YZ. These names have to do with the orientation of a dynamo-generated magnetic field. These dynamo-generated magnetic fields take the form of Beltrami fields that vary in the \( x \), \( y \), and \( z \) directions for branches X, Y, and Z, while for branch YZ the field varies both in the \( y \) and \( z \) directions. Earlier work without imposed fields has shown that branch YZ can be accessed during intermediate times during the saturation of the dynamo, but it is not one of the ultimate stable branches X, Y, or Z.

Branches Y and Z show the sudden onset of suppression of \( \alpha_{\text{imp}} \) for weak magnetic fields. This has to do with the fact that for weak imposed magnetic fields a dynamo-generated field of Beltrami type is being generated. Such fields quench the \( \alpha \) effect, even though they do not contribute to the volume-averaged mean field. On branch YZ the \( \alpha \) effect is only weakly suppressed, while on branch X the imposed field \( \alpha_{\text{imp}} \) increases with decreasing values of \( B_0 \).

The test-field method reveals that on branches X, Y, and YZ the \( \alpha_{yy} \) component is nearly independent of \( B_0 \), and always larger than the \( \alpha_{xx} \) component. However, on branch Z and for \( R_m B_0^2/B_{eq}^2 < 1 \) we find that \( \alpha_{xx} = \alpha_{yy} \) and only weakly suppressed.

A comment regarding the discontinuities in Fig. 3 near \( R_m B_0^2/B_{eq}^2 = 1 \) is here in order. The systems considered here are in saturated states. To the left of the discontinuities the system has a saturated meso-scale dynamo, while to the right there is none.

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Intermediate states are simply not possible. Hence, the discontinuities are caused by the effects of the meso-scale magnetic fields on \( u_{\text{mix}} \) and thus on \( R_{\alpha} \).

3.2 Relation to \( \alpha_1 \) and \( \alpha_2 \)

In the following we will try to interpret the results presented above in terms of equation (11) and determine \( \alpha_1 \) and \( \alpha_2 \) for the different branches. For small values of \( B_0 \), a magnetic field with \( k = k_1 \) and hence a finite planar average can develop. Compared with the large-scale field \( B_0 \), we refer to this dynamo-generated field as meso-scale magnetic field. As demonstrated in Brandenburg (2001), three types of such mean fields are possible in the final saturated state. These fields correspond to Beltrami fields of the form

\[
\frac{\mathbf{B}^{(x)}}{B_1} = \begin{pmatrix} 0 \\ c_{x} \\ s_{x} \end{pmatrix}, \quad \frac{\mathbf{B}^{(y)}}{B_1} = \begin{pmatrix} s_{y} \\ 0 \\ c_{y} \end{pmatrix}, \quad \frac{\mathbf{B}^{(z)}}{B_1} = \begin{pmatrix} c_{z} \\ s_{z} \\ 0 \end{pmatrix},
\]

where \( c_{x} = \cos (k_1 \xi + \phi) \) and \( s_{x} = \sin (k_1 \xi + \phi) \) denote cosine and sine functions as functions of \( \xi = x, y \) or \( z \), with an arbitrary phase shift \( \phi \).

The precise value of \( k_1 \) is quenched by \( k_0 \) and \( c_{\alpha} \), so we can calculate

\[
\tilde{\alpha}_1 = \tilde{\alpha}_{xy}, \quad \tilde{\alpha}_2 = \tilde{\alpha}_{xy} - \tilde{\alpha}_{yy},
\]

where a tilde indicates normalization by \( \alpha_0 \). For weak imposed fields, \( \beta \to 0 \), we can calculate \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) on the X branch by using the relations

\[
\tilde{\alpha}_{xx} = \tilde{\alpha}_1, \quad \tilde{\alpha}_{yy} = \tilde{\alpha}_1 + \frac{1}{2} \tilde{\alpha}_2, \quad \tilde{\alpha}_{xy} = \tilde{\alpha}_1 + \tilde{\alpha}_2.
\]

However, on the X branch \( \tilde{\alpha}_{xx} \) is ill determined, as seen in Fig. 3 and discussed in Section 4.1. Therefore, we use only equations (19) and (20) to calculate

\[
\tilde{\alpha}_1 = 2\tilde{\alpha}_{xy} - \tilde{\alpha}_{xy}, \quad \tilde{\alpha}_2 = 2\tilde{\alpha}_{xy} - 2\tilde{\alpha}_{yy},\]

For the Y, Z and YZ branches, on the other hand, these relations have to be substituted by

\[
\tilde{\alpha}_1 = 2\tilde{\alpha}_{xx} - \tilde{\alpha}_{xy}, \quad \tilde{\alpha}_2 = 2\tilde{\alpha}_{xy} - 2\tilde{\alpha}_{xx}.
\]

Figure 4. Root-mean-square values of the mean magnetic fields as functions of the imposed field for turbulence with \( R_{\alpha} = 26 \) for the X, Y, Z and YZ branches in the same order as in Fig. 3. Diamonds, triangles and squares denote \( \mathbf{B}^{(x)}, \mathbf{B}^{(y)} \) and \( \mathbf{B}^{(z)} \), respectively.

The resulting values of \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) are plotted in Fig. 5 for each of the four branches. On the Y branch one can, as a test, also use the independent relation \( \tilde{\alpha}_1 = \tilde{\alpha}_{yy} \). The resulting values are about 50 per cent larger than the values shown in Fig. 5, suggesting that there could be additional contributions in the simplified relation \( \tilde{\alpha}_{yy} = \tilde{\alpha}_1 \). On the Z branch, of course, \( \tilde{\alpha}_{xx} = \tilde{\alpha}_{yy} \), so here too we have to use the equations (22).

In all cases we find that \( \tilde{\alpha} \) is quenched by \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) having opposite signs and their moduli approaching each other. This is particularly clear in the case of strong fields where \( \tilde{\alpha}_1 \) and \( -\tilde{\alpha}_2 \) become indistinguishable, while each of them is still increasing. We note that the turbulence itself is not strongly affected (Brandenburg & Subramanian 2005a). On the Y and Z branches both \( \tilde{\alpha}_1 \) and \( \tilde{\alpha}_2 \) are of order unity, but on the X branch they can reach rather large values when the imposed field is weak. The behaviour on the YZ branch is somewhat unsystematic, suggesting that this branch is really just the result of a long-term transient, as was already found in the absence of an imposed field (Brandenburg 2001). However, we decided not

\[1\] Unlike the case considered by Brandenburg et al. (2008b), here the test field has \( k = 0 \), and there is no relative phase to be considered.

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to discard this branch, because it is likely that transient solutions on this branch may become even more long-lived as the magnetic Reynolds number is increased further.

3.3 Enhancement of $\alpha_{imp}$ in the field-aligned case

The suppression of $\alpha = \alpha_1 + \alpha_2$ by the magnetic field is not surprising. What is unexpected, however, is the dramatic enhancement of both $\alpha_1$ and $-\alpha_2$ for weak imposed fields and equipartition-strength meso-scale fields that vary in the $x$ direction (the field-aligned case or X branch). In this case the interactions of the current density associated with the Beltrami field and the imposed field generate a force varying along $x$, perpendicular to the components of the meso-scale Beltrami field. This generates a meso-scale velocity that in turn damps the Beltrami field, resulting in the slower rise in $B_{eq}$ as $B_0/B_{eq}$ is decreased. Further, the cross-product of the meso-scale velocity field with the Beltrami field generates a large-scale electromotive force in the $x$ direction. This is seen both in $\alpha_{imp}$ and in $\alpha_{zz}$. A rough estimate of this electromotive force can be obtained by considering the fields

$$B_0 = \begin{pmatrix} B_0 \\ 0 \\ 0 \end{pmatrix}, \quad B_1 = B_1 \begin{pmatrix} 0 \\ \cos kx \\ -\sin kx \end{pmatrix},$$

so that $\mu_0 J_1 = -k B_1$, where subscript 1 denotes meso-scale fields. The meso-scale current density and the imposed field will generate a meso-scale Lorentz force which will drive a meso-scale velocity field $U_1$. We estimate $U_1$ by balancing

$$J_1 \times B_0/\rho + v_1 V^2 U_1 \approx 0,$$

where $v_1$ is the turbulent viscosity. We therefore expect $U_1$ will saturate for

$$U_1 = B_0 B_1/\rho \mu_0 \begin{pmatrix} 0 \\ \sin kx \\ -\cos kx \end{pmatrix}.$$

This velocity field will generate an $E_0$ parallel to $B_0$ in conjunction with $B_1$:

$$E_0 \equiv \langle U_1 \times B_1 \rangle = \alpha_{meso} B_0,$$

with $\alpha_{meso} = B_1^2/(\rho \mu_0 v_1 k)$. We then expect the total $\alpha_{imp}$ to be

$$\alpha_{imp} = \alpha + B_1^2/\rho \mu_0 v_1 k.$$

Normalizing by $\alpha_0 = -u_{rms}/3$ and assuming $v_1 \approx u_{rms}/3 k_1$ we find for small imposed field and a meso-scale dynamo that varies along $x$:

$$\frac{\alpha_{imp}}{\alpha_0} \approx 1 + 9 \frac{k_1}{k} \left( \frac{B_1}{B_{eq}} \right)^2.$$

Given that $k_1/k = 3$ and noting that $B_1/B_{eq}$ reaches values up to 1.2, we find that $\alpha_{imp}/\alpha_0 \approx 40$, which is still somewhat below the actual value of 53; see the top panel of Fig. 3. The remaining discrepancy may be explicable by recalling that the actual value of $v_1$ may well be reduced due to the presence of an equipartition-strength magnetic field.

3.4 Comment on wavenumber dependence

In previous work on the test-field method we used test fields with wavenumbers different from zero. It turned out that in the kinematic regime, $\alpha$ is proportional to $1/(1 + a(k/k_1)^2)$, where $a = 0.5, \ldots, 1$ (Brandenburg et al. 2008a; Mitra et al. 2009). It was shown that the variation of $\alpha$ with $k$ represents non-locality in space. In order to get some idea about the dependence of $\alpha_{xx}$ and $\alpha_{yy}$ on $k$ in the present case we compare in Table 3 the results for $k = k_0$ with those for $k = 0$. It turns out that both values decrease by 30 per cent on the X branch, and increase by less than 10 per cent on the Z branch.

Table 3. Examples of the dependence of $\alpha_{xx}$ and $\alpha_{yy}$ on the wavenumber $k$ of the test field. Note that the field strength is different in both cases.

<table>
<thead>
<tr>
<th>Branch</th>
<th>$k/k_0$</th>
<th>$\alpha_{xx}$</th>
<th>$\alpha_{yy}$</th>
<th>$B_{imp}^{1/2} B_0/B_{eq}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>0</td>
<td>0.72 ± 0.14</td>
<td>0.51 ± 0.16</td>
<td>0.06</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.61 ± 0.02</td>
<td>0.37 ± 0.01</td>
<td>0.06</td>
</tr>
<tr>
<td>Z</td>
<td>0</td>
<td>0.34 ± 0.02</td>
<td>0.32 ± 0.02</td>
<td>0.2</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.35 ± 0.01</td>
<td>0.35 ± 0.02</td>
<td>0.2</td>
</tr>
</tbody>
</table>
The $k$ dependence for the Z branch is minor, although one would have expected a small decrease rather than an increase. Nevertheless, within error bars, this result is possibly still compatible with the dependence in the kinematic case. For the X branch the error bars for $k = 0$ are larger. This is because of the strong interaction between the imposed uniform field and a Beltrami field varying along the same direction, as discussed in Section 3.3. It is therefore not clear whether the $k$ dependence is here significant and how to interpret it.

4 RESETTING THE FLUCTUATIONS

4.1 Effectiveness of resetting the fields

The evolution equations used both in the imposed-field method and in the test-field method allow for dynamo action. This led Ossendrijver et al. (2002) and Käpylä et al. (2006) to the technique of resetting the resulting magnetic field in regular intervals. This method is now also routinely used in the test-field approach (Sur et al. 2008), and we have also used it throughout this work. The lack of resetting the magnetic field may also be the main reason for the rather low values of $\alpha$ found in the recent work of Hughes & Proctor (2009); see the corresponding discussion in Käpylä et al. (2009b).

In this section we employ the method of resetting $B$ to obtain better estimates for $\alpha$ for weak imposed fields, and to compare this with results from the test-field method. The result is shown in Fig. 6 where we show the dependence of $\alpha_{\text{imp}}$ on $B_0$ and on the reset interval $\Delta t$. We note that, in units of the turnover time, the reset interval $\Delta t u_{\text{rms}} k_1$ has a weak dependence both on $B_0$ and $\Delta t$.

![Figure 6](image)

**Figure 6.** Dependence of $\alpha_{\text{imp}}$ (solid lines) and $\alpha_{\text{imp}}$ (dotted lines) on the imposed field strength with fixed reset time $\Delta t u_{\text{rms}} k_1 = 50, \ldots, 70$ (upper panel) and the dependence of $\alpha_{\text{imp}}$ on the reset time for $R_{\text{m}}^{1/2} B_0/B_{\text{eq}} = 0.1$ (lower panel). In all cases we have $R_{\text{m}} \approx 30$.

Table 4. Comparison of the results for $\alpha_{xx}$ and $\alpha_{yy}$ for two different reset times $\Delta t$ for the examples of the X and Z branches with $R_{\text{m}}^{1/2} B_0/B_{\text{eq}} = 0.2$. The reset time is normalized by the inverse turnover time $(u_{\text{rms}} k_1)^{-1}$.

<table>
<thead>
<tr>
<th>Branch</th>
<th>$\Delta t u_{\text{rms}} k_1$</th>
<th>$\alpha_{xx}$</th>
<th>$\alpha_{yy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>25</td>
<td>$0.08 \pm 0.13$</td>
<td>$0.54 \pm 0.02$</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>$0.98 \pm 0.09$</td>
<td>$0.70 \pm 0.04$</td>
</tr>
<tr>
<td>Z</td>
<td>25</td>
<td>$0.34 \pm 0.02$</td>
<td>$0.32 \pm 0.02$</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>$0.32 \pm 0.01$</td>
<td>$0.33 \pm 0.03$</td>
</tr>
</tbody>
</table>

because small values of $B_0$ and $\Delta t$ quench $\alpha_{\text{imp}}$ only weakly. The resetting technique has eliminated the branching for weak fields. For weak fields we find that the value of $\alpha_{\text{imp}}$ is slightly below $\alpha_{\text{eq}}$, but this is partly because for finite scale separation there is an additional factor $(1 + k_x^2/k_y^2)^{-1} \approx 0.9$ (Brandenburg et al. 2008a). The actual value of $\alpha_{\text{imp}}$ is somewhat smaller still, which may be ascribed to other systematic effects.

It turns out that over a wide range of reset intervals the resulting values of $\alpha_{\text{imp}}$ are not dependent in a systematic way on the reset interval (see also Mitra et al. 2009), although it is clear that the error bars increase for larger values of $\Delta t$. The same is true for the values of $\alpha_{xx}$ and $\alpha_{yy}$ obtained using the test-field method, except for the case of weak fields on the X branch where the values of $\alpha_{xx}$ are ill determined; see Table 4, where we compare the values of $\alpha_{xx}$ and $\alpha_{yy}$ for two different reset times in the case where $\alpha_{xx}$ is found to change sign ($R_{\text{m}}^{1/2} B_0/B_{\text{eq}} \approx 0.2$). The increasing fluctuations for longer reset intervals occur as the system exits the kinematic regime. It might therefore be possible to find indicators of when the kinematic regime has been exited and resetting becomes necessary. However, we have not pursued this further in this work.

For even larger values of $\Delta t$ there is enough time for the mesoscale magnetic field to develop. An example is shown in Fig. 7 where 18 intervals of length $\Delta t u_{\text{rms}} k_1 = 270$ are shown. For half of these intervals the wavevector of the Beltrami field begins to develop in the $x$ direction, so $\alpha_{\text{imp}}$ is heading toward the X branch. In the other half of these cases the magnetic field is weak and $\alpha_{\text{imp}}$ lies on one of the other branches. None of these cases reproduce the correct kinematic value of $\alpha$, because we are not really considering a kinematic problem in this case. This underlines the importance of choosing reset intervals that are not too long.

Our results support the hypothesis that the precise value of the reset time interval is not critical except for the field-aligned case.
where the diagonal components of the $\alpha_{ij}$ tensor are large and quite uncertain, as indicated also by the large error bars. The sign change found for $\alpha_{xx}$ at low or intermediate field strengths might therefore not be real.

### 4.2 Time averaging in the test-field method

We have already demonstrated that the length of the reset interval is not critical for the value of $\alpha$, but longer reset times tend to lead to larger errors. In the present section we demonstrate this for the test-field method using the idealized case where the turbulent flow velocity is replaced by simple stationary flow given by the equation

$$ U = k_0 \varphi \hat{z} + \nabla \times (\varphi \hat{z}), $$

with

$$ \varphi = \varphi(x, y) = \mu_0 \cos k_0 x \cos k_0 y, $$

which is known as the Roberts flow.

When the magnetic Reynolds number exceeds a critical value of around 60, some kind of dynamo action of $b^0$ commences. This type of dynamo is often referred to as small-scale dynamo action (Brandenburg et al. 2008b; Sur et al. 2008; Cattaneo & Hughes 2009), but this name may not always be accurate. In the case of the Roberts flow there would be no such dynamo action if the wavenumber of the test field is zero, $k = 0$, as assumed here. However, for $k = k_0$, for example, dynamo action for the test-field equation is possible. The test fields are therefore chosen to be

\[
\begin{align*}
\frac{\mathbf{B}}{B} &= \begin{pmatrix} \cos kz \\ -sin kz \\ 0 \end{pmatrix}, \\
\frac{\mathbf{B}^0}{B} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
\frac{\mathbf{B}^\perp}{B} &= \begin{pmatrix} 0 \\ 0 \\ \sin kz \end{pmatrix}, \\
\frac{\mathbf{B}^\parallel}{B} &= \begin{pmatrix} \cos kz \\ 0 \\ 0 \end{pmatrix},
\end{align*}
\]

see Sur et al. (2008). Since now the mean fields are also functions of $z$, the term $u \times \mathbf{B}^0$ cannot be omitted in equation (8).

As stressed by Brandenburg et al. (2008a), in the expression for the electromotive force there is in general also a contribution $\mathbf{E}_0$ that is independent of the mean field. Given that test fields $\mathbf{B}^0$ are independent of time, we have

\[
\mathbf{E}'(z, t) = \mathbf{E}'_0(z, t) + \alpha(z)\mathbf{B}^\perp - \eta_i(z)\mu_0 \mathbf{J}(z),
\]

where overbars denote $xy$ averages (not volume averages), so there is also a term $\eta_i \mu_0 \mathbf{J}$, where $\eta_i$ is the turbulent magnetic diffusivity. We have assumed that $\alpha$ and $\eta_i$ are independent of time, and in this case they are also independent of $z$. The $\mathbf{E}'_0(z, t)$ term can be eliminated by averaging over time, i.e. $\langle \mathbf{E}'_0 \rangle = 0$, so

\[
\langle \mathbf{E}' \rangle = \alpha \mathbf{B}^\perp - \eta_i \mu_0 \mathbf{J}.
\]

In Fig. 8 we show the evolution of $\alpha$ for the Roberts flow with $R_m = 65$ and 55. In the case with $R_m = 65$ there are exponentially growing oscillations corresponding to a wave travelling in the $z$ direction. In general such fields can be a superposition of waves travelling in the positive and negative $z$ directions. It is seen quite clearly that the running time average is stable and well defined. The results for $R_m = 65$ and 55 are close together ($\alpha/\alpha_0 = 0.096$ and 0.090, respectively), suggesting continuity across the point where dynamo action sets in. This supports the notion that averaging over time is a meaningful procedure.

### 5 Conclusions

The present simulations have shown that the imposed-field method leads to a number of interesting and unexpected results. For imposed fields exceeding the value $R_m^{1/2} B_{eq}$ one recovers the catastrophic quenching formula of Vainshtein & Cattaneo (1992); see equation (15). We emphasize once more, however, that this formula is only valid for completely uniform large-scale fields in a triply periodic domain. This is clearly artificial, but it provides an important benchmark.

A number of surprising results have been found for weaker fields of less than $R_{m1}^{1/2} B_{eq}$. In virtually none of those cases does the imposed-field method recover the kinematic value of $\alpha$. Instead, $\alpha_{sup}$ can attain strongly suppressed values, but it can actually also attain strongly enhanced values. This is caused by the unavoidable emergence of meso-scale dynamo action. In principle, such meso-scale dynamo action could have been suppressed by restricting oneself to scale-separation ratios, $k_z/k_1$, of less than 2 or so. This was done, for example, in some of the runs of Brandenburg & Subramanian (2005a). In the present case of a triply periodic box, four different magnetic field configurations can emerge. The first three correspond to Beltrami fields, where the wavevector points in one of the three coordinate directions. The fourth possibility is also a Beltrami field, but one that varies diagonally in a direction perpendicular to the direction of the imposed field. The latter was found to be unstable in the absence of an imposed field, but they can be long-lived in the present case of an imposed field.

In this paper, we have used the term meso-scale fields to refer to the Beltrami fields naturally generated by the helicity-driven dynamo in our system. A more general definition of meso-scale fields would encompass all fields that break isotropy, average to zero, and yet do not time-average to zero. In the absence of such fields, mean-field theory can be applied in a straightforward manner. This is indeed the case that one is normally interested in. However, when
such meso-scale fields exist, they must be understood for determining turbulent transport coefficients, because those coefficients apply then to the particular case of saturated meso-scale fields.

The results obtained with the imposed-field method reflect correctly the circumstances in the non-linear case where the $\alpha$ effect is suppressed by dynamo-generated meso-scale magnetic fields whose scale is smaller than that of the imposed field, but comparable to the scale of the domain. Especially in the case of closed or periodic domains the resulting $\alpha$ is catastrophically quenched, which is now well understood (Blackman & Brandenburg 2002; Field & Blackman 2002). This effect is particularly strong in the case where one considers volume averages, and thus ignores the effects of turbulent magnetic diffusion. With magnetic diffusion included, both $\alpha$ and $\eta_t$ have only a mild dependence on $R_m$ (Brandenburg et al. 2008b). However, astrophysical dynamos are expected to operate in a regime where magnetic helicity fluxes alleviate catastrophic quenching; see Brandenburg & Subramanian (2005b) for a review.

Determining the nature of the dynamo mechanism is an important part in the analysis of a successful simulation showing large-scale field generation. Our present analysis shows that meaningful results for $\alpha$ can be obtained using either the imposed-field or the test-field methods provided the departure of the magnetic field from $B_0$ is reset to zero to eliminate the effects of dynamo-generated meso-scale magnetic fields. Conversely, if such fields are not eliminated, the results can still be meaningful, as demonstrated here, but they need to be interpreted correspondingly and bear little relation to the imposed field. On the other hand, for strong imposed magnetic fields ($R_m B_0^2/\eta_t^2 > 1$), meso-scale magnetic fields tend not to grow, so the resetting procedure is then neither necessary nor would it make much of a difference when the test-field method is used. However, when the imposed-field method is used, the resetting of the actual field reduces the quenching of $a_{\text{mean}}$. This affects the normalizations of $B_0$ and $\alpha_0$ with $\eta_t$, and $\alpha_0$, respectively, because both are proportional to $a_{\text{mean}}$.

Throughout this paper we have considered relatively moderate values of $R_m$, but we computed a large number of different simulations. In the beginning of this study we started with larger values of $R_m$ and found that the resulting $a_{\text{map}}$ seemed inconsistent. In hindsight it is clear what happened: the few cases that we had in the beginning were all scattered around different branches. Only later, by performing a large number of simulations at smaller values of $R_m$ it became clear that there are indeed different branches. This highlights the importance of studying not just one or a few models of large $R_m$, but rather a larger systematic set of intermediate cases of moderate $R_m$ where it is possible to understand in detail what is going on. It will be important to continue exploring the regime of larger $R_m$, and we hope that the new understanding that emerged from studying cases of moderate $R_m$ proves useful in this connection.

According to the results available so far, we can say that for larger values of $R_m$ the turbulent transport coefficients are only weakly affected (see Brandenburg et al. 2008b, for $R_m \leq 600$) for fields of equipartition strength, or not affected at all (Sur et al. 2008, for $R_m \leq 220$) if the field is in the kinematic limit.

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Magnetic-field decay of three interlocked flux rings with zero linking number

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The resistive decay of chains of three interlocked magnetic flux rings is considered. Depending on the relative orientation of the magnetic field in the three rings, the late-time decay can be either fast or slow. Thus, the qualitative degree of tangledness is less important than the actual value of the linking number or, equivalently, the net magnetic helicity. Our results do not suggest that invariants of higher order than that of the magnetic helicity need to be considered to characterize the decay of the field.

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I. INTRODUCTION

Magnetic helicity plays an important role in plasma physics [1–3], solar physics [4–6], cosmolgy [7–9], and dynamo theory [10,11]. This is connected with the fact that magnetic helicity is a conserved quantity in ideal magnetohydrodynamics [12]. The conservation law of magnetic helicity is ultimately responsible for inverse cascade behavior that can be relevant for spreading primordial magnetic field over large length scales. It is also likely the reason why the magnetic fields of many astrophysical bodies have length scales that are larger than those of the turbulent motions responsible for driving these fields. In the presence of finite magnetic diffusivity, the magnetic helicity can only change on a resistive time scale. Of course, astrophysical bodies are open, so magnetic helicity can change by magnetic helicity fluxes out of or into the domain of interest. However, such cases will not be considered in the present paper.

In a closed or periodic domain without external energy supply, the decay of a magnetic field depends critically on the value of the magnetic helicity. This is best seen by considering spectra of magnetic energy and magnetic helicity. The magnetic energy spectrum $M(k)$ is normalized such that

$$\int M(k) dk = (B^2)/2\mu_0,$$  \hspace{1cm} (1)

where $B$ is the magnetic field, $\mu_0$ is the magnetic permeability, and $k$ is the wave number (ranging from 0 to $\infty$). The magnetic helicity spectrum $H(k)$ is normalized such that

$$\int H(k) dk = (A \cdot B),$$  \hspace{1cm} (2)

where $A$ is the magnetic vector potential with $B = \nabla \times A$. In a closed or periodic domain, $H(k)$ is gauge invariant, i.e., it does not change after adding a gradient term to $A$. For finite magnetic helicity, the magnetic energy spectrum is bound from below [12] such that

$$M(k) \geq kH(k)/2\mu_0.$$  \hspace{1cm} (3)

This relation is also known as the realizability condition [13]. Thus, the decay of a magnetic field is subject to a corresponding decay of its associated magnetic helicity. Given that in a closed or periodic domain the magnetic helicity changes only on resistive time scales [14], the decay of magnetic energy is slowed down correspondingly. More detailed statements can be made about the decay of turbulent magnetic fields, where the energy decays in a power-law fashion proportional to $r^{-\sigma}$. In the absence of magnetic helicity, $(A \cdot B)=0$, we have a relatively rapid decay with $\sigma \approx 1.3$ [15], while with $(A \cdot B) \neq 0$, the decay is slower with $\sigma$ between 1/2 [9] and 2/3 [16].

The fact that the decay is slowed down in the helical case is easily explained in terms of the topological interpretation of magnetic helicity. It is well known that the magnetic helicity can be expressed in terms of the linking number $n$ of discrete magnetic flux ropes via [13]

$$\int A \cdot B dV = 2\pi \Phi_1 \Phi_2,$$  \hspace{1cm} (4)

where

$$\Phi_i = \int_S B \cdot dS \quad (\text{for } i = 1 \text{ and } 2)$$  \hspace{1cm} (5)

are the magnetic fluxes of the two ropes with cross-sectional areas $S_1$ and $S_2$. The slowing down of the decay is then plausibly explained by the fact that a decay of magnetic energy is connected with a decay of magnetic helicity via the realizability condition (3). Thus, a decay of magnetic helicity can be achieved either by a decay of the magnetic flux or by magnetic reconnection. Magnetic flux can decay through annihilation with oppositely oriented flux. Reconnection on the other hand reflects a change in the topological connectivity, as demonstrated in detail in Ref. [17], p. 28.

The situation becomes more interesting when we consider a flux configuration that is interlocked, but with zero linking number. This can be realized quite easily by considering a configuration of two interlocked flux rings where a third flux ring is connected with one of the other two rings such that the total linking number becomes either 0 or 2, depending on the relative orientation of the additional ring, as is illustrated in Fig. 1. Topologically, the configuration with linking numbers of 0 and 2 are the same except that the orientation of the field lines in the upper ring is reversed. Nevertheless, the simple topological interpretation becomes problematic in the case of zero linking number, because then also the magnetic helicity is zero, so the bound of $M$ from below disappears,
and $M$ can now in principle freely decay to zero. One might expect that the topology should then still be preserved and that the linking number as defined above, which is a quadratic invariant, should be replaced with a higher-order invariant [18–20]. It is also possible that in a topologically interlocked configuration with zero linking number the magnetic helicity spectrum $H(k)$ is still finite and that bound (3) may still be meaningful. In order to address these questions we perform numerical simulations of the resistive magnetohydrodynamic equations using simple interlocked flux configurations as initial conditions. We also perform a control run with a noninterlocked configuration and zero helicity in order to compare the magnetic energy decay with the interlocked case.

Magnetic helicity evolution is independent of the equation of state and applies hence to both compressible and incompressible cases. In agreement with earlier work [21] we assume an isothermal gas, where pressure is proportional to density and the sound speed is constant. However, in all cases the bulk motions stay subsonic, so for all practical purposes our calculations can be considered nearly incompressible, which would be an alternative assumption that is commonly made [22].

II. MODEL

We perform simulations of the resistive magnetohydrodynamic equations for a compressible isothermal gas where the pressure is given by $p = \rho c_s^2$, with $\rho$ being the density and $c_s$ being the isothermal sound speed. We solve the equations for $A$, the velocity $U$, and the logarithmic density $\ln \rho$ in the form

$$\frac{dA}{dt} = U \times B + \eta \nabla^2 A,$$

(6)

$$\frac{DU}{Dt} = -c_s^2 \nabla \ln \rho + J \times B / \rho + F_{\text{visc}},$$

(7)

where $F_{\text{visc}} = \rho^{-1} \nabla \cdot 2 \nu \rho S$ is the viscous force; $S$ is the traceless rate of strain tensor, with components $S_{ij} = \frac{1}{2}(U_{ij} + U_{ji}) - \frac{1}{3} \delta_{ij} \nabla \cdot U$; $J = \nabla \times B / \mu_0$ is the current density; $\nu$ is the kinematic viscosity; and $\eta$ is the magnetic diffusivity.

The initial magnetic field is given by a suitable arrangement of magnetic flux ropes, as already illustrated in Fig. 1. These ropes have a smooth Gaussian cross-sectional profile that can easily be implemented in terms of the magnetic vector potential. We use the PENCIL code [23], where this initial condition for $A$ is already prepared, except that now we adopt a configuration consisting of three interlocked flux rings (Fig. 1) where the linking number can be chosen to be either 0 or 2, depending only on the field orientation in the last (or the first) of the three rings. Here, the two outer rings have radii $R_o$, while the inner ring is slightly bigger and has the radius $R_i = 1.2 R_o$, but with the same flux. We use $R_o$ as our unit of length. The sound travel time is given by $T_s = R_o / c_s$.

In the initial state we have $U = 0$ and $\rho = \rho_0 = 1$. Our initial flux, $\Phi = \int B \cdot dS$, is the same for all tubes with $\Phi = 0.1 c_i R_i^2 \mu_0 \rho_0$. This is small enough for compressibility effects to be unimportant, so the subsequent time evolution is not strongly affected by this choice. For this reason, the Alfvén time, $T_\alpha = \sqrt{\mu_0 \rho_0 R_i^2 / \Phi}$, will be used as our time unit. In all our cases we have $T_\alpha = 10 T_s$ and denote the dimensionless time as $\tau = t / T_\alpha$. In all cases we assume that the magnetic Prandtl number $\nu / \eta$ is unity, and we choose $\nu = \eta = 10^{-4} R_c c_s = 10^{-2} R_i^2 / T_\alpha$. We use $256^3$ mesh points.

We have chosen a fully compressible code, because it is readily available to us. Alternatively, as discussed at the end of Sec. I, one could have chosen an incompressible code by ignoring the continuity equation and computing the pressure such that $\nabla \cdot U = 0$ at all times. Such an operation breaks the locality of the physics and is computationally more intensive, because it requires global communication.

III. RESULTS

Let us first discuss the visual appearance of the three interlocked flux rings at different times. In Fig. 2 we compare the three rings for the zero and finite magnetic helicity cases at the initial time and at $\tau = 0.5$. Note that each ring shrinks as a result of the tension force. This effect is strongest in the core of each ring, causing the rings to show a characteristic indentation that was also seen in earlier inviscid and nonresistive simulations of two interlocked flux rings [21].

At early times, visualizations of the field show little difference, but at time $\tau = 0.5$ some differences emerge in that the configuration with zero linking number develops an outer ring encompassing the two rings that are connected via the inner ring; see Fig. 2. This outer ring is absent in the configuration with finite linking number.

The change in topology becomes somewhat clearer if we plot the magnetic-field lines (see Fig. 3). For the $n=2$ configuration, at time $\tau=4$ one can still see a structure of three interlocked rings, while for the $n=0$ case no clear structure...
can be recognized. Note that the magnitude of the magnetic field has diminished more strongly for $n=0$ than for $n=2$. This is in accordance with our initial expectations.

The differences between the two configurations become harder to interpret at later times. Therefore, we compare in Fig. 4 cross sections of the magnetic field for the two cases. The $xy$ cross sections show clearly the development of the new outer ring in the zero linking number configuration. From this figure it is also evident that the zero linking number case suffers more rapid decay because of the now anti-aligned magnetic fields (in the upper panel $B_z$ is of opposite sign about the plane $y=0$ while it is negative in the lower panel).

The evolution of magnetic energy is shown in Fig. 5 for the cases with zero and finite linking numbers. Even at the time $\tau=0.6$, when the rings have just come into mutual contact, there is no clear difference in the decay for the two cases. Indeed, until the time $\tau=2$ the magnetic energy evolves still similarly in the two cases, but then there is a pronounced difference where the energy in the zero linking number case shows a rapid decline (approximately like $t^{-3.2}$), while in the case with finite linking number it declines much more slowly (approximately like $t^{-1.3}$). However, power-law behavior is only expected under turbulent conditions and not for the relatively structured field configurations considered here. The energy decay in the zero linking number case is roughly the same as in a case of three flux rings that are not interlocked. The result of a corresponding control run is shown as a dotted line in Fig. 5. At intermediate times, $0.5 < \tau< 5$, the magnetic energy of the control run has diminished somewhat faster than in the interlocked case with $n=0$. It is possible that this is connected with the interlocked nature of the flux rings in one of the cases. Alternatively, this might reflect the presence of rather different dynamics in the noninterlocked case, which seems to be strongly controlled by oscillations on the Alfvén time scale. Nevertheless, at later times the decay laws are roughly the same for noninterlocked and interlocked nonhelical cases.

The time when the rings come into mutual contact is marked by a maximum in the kinetic energy at $\tau=0.6$. This can be seen from Fig. 6, where we compare kinetic and magnetic energies separately for the cases with finite and zero linking numbers. Note also that in the zero linking number case magnetic and kinetic energies are nearly equal and decay in the same fashion.

Next we consider the evolution of magnetic helicity in Fig. 7. Until the time $\tau=0.6$ the value of the magnetic helicity has hardly changed at all. After that time there is a gradual decline, but it is slower than the decline of magnetic energy. Indeed, the ratio $(\mathbf{A} \cdot \mathbf{B})/|\mathbf{B}|^2$, which corresponds to a length scale, shows a gradual increase from $0.1R_o$ to nearly $0.6R_o$ at the end of the simulation. This reflects the fact that the field has become smoother and more space filling with time.

Given that the magnetic helicity decays only rather slowly, one must expect that the fluxes $\Phi_i$ of the three rings also only change very little. Except for simple configurations where flux tubes are embedded in field-free regions, it is in general difficult to measure the actual fluxes, as defined in Eq. (5). On the other hand, especially in observational solar physics, one often uses the so-called unsigned flux [24,25], which is defined as...
For a ring of flux $\Phi$ that intersects the surface in the middle at right angles the net flux cancels to zero, but the unsigned flux gets contributions from both intersections, so $P_{2D} = 2 |\Phi|$. In three-dimensional simulations it is convenient to determine

$$P_{2D} = \int_S |\mathbf{B}| dS.$$  \hspace{1cm} (9)

FIG. 4. (Color online) Cross sections in the $xy$ plane of the magnetic field with zero linking number (upper row) and finite linking number (lower row). The $z$ component (pointing out of the plane) is shown together with vectors of the field in the plane. Light (yellow) shades indicate positive values and dark (blue) shades indicate negative values. Intermediate (red) shades indicate zero value.

FIG. 5. Decay of magnetic energy (normalized to the initial value) for linking numbers of 2 (solid line) and 0 (dashed line). The dotted line gives the decay for a control run with noninterlocked rings. The dashed-dotted lines indicate $t^{1/4}$ and $t^{3/2}$ scalings for comparison. The inset shows the evolution of the maximum field strength in units of the thermal equipartition value, $B_{th} = c_s (\rho_0 \mu_0)^{1/2}$.

FIG. 6. Comparison of the evolution of kinetic and magnetic energies in the cases with finite and with vanishing linking numbers. Note that in both cases the maximum kinetic energy is reached at the time $\tau = 0.6$. The two cases begin to depart from each other after $\tau = 2$. In the nonhelical case the magnetic energy shows a sharp drop and reaches equipartition with the kinetic energy, while in the helical case the magnetic energy stays always above the equipartition value.
\[ P = \int_V |\mathbf{B}| dV. \] (10)

For several rings, all with radius \( R \), we have
\[ P = 2\pi R \sum_{i=1}^N |\Phi_i| = \pi NR_{\text{PD}}, \] (11)

where \( N \) is the number of rings. In Fig. 8 we compare the evolution of \( P \) (normalized to the initial value \( P_0 \)) for the cases with \( n=0 \) and \( 2 \). It turns out that after \( \tau=1 \) the value of \( P \) is nearly constant for \( n=2 \), but not for \( n=0 \).

Let us now return to the earlier question of whether a flux configuration with zero linking number can have finite spectral magnetic helicity, i.e., whether \( H(k) \) is finite but of opposite sign at different values of \( k \). The spectra \( M(k) \) and \( k|H(k)|/2\mu_0 \) are shown in Fig. 9 for the two cases at time \( \tau=5 \). This figure shows that in the configuration with zero linking number \( H(k) \) is essentially zero for all values of \( k \). This is not the case and, in hindsight, is hardly expected; see Fig. 9 for the spectra of \( M(k) \) and \( k|H(k)|/2\mu_0 \) in the two cases at \( \tau=5 \). What might have been expected is a segregation of helicity not in the wave-number space, but in the physical space for positive and negative values of \( y \). It is then possible that magnetic helicity has been destroyed by locally generated magnetic helicity fluxes between the two domains in \( y>0 \) and \( y<0 \). However, this is not pursued further in this paper.

In order to understand in more detail the way the energy is dissipated, we plot in Fig. 10 the evolution of the time derivative of the magnetic energy \( E_M=\frac{1}{2} \int_V \mathbf{B}^2 dV \) (upper panel) and the kinetic energy \( E_K=\frac{1}{2} \int_V \mathbf{U}^2 dV \) (lower panel). In the lower panel we also show the rate of work done by the Lorentz force, \( W_L = \int_V (\mathbf{U} \cdot \mathbf{J} \times \mathbf{B}) dV \), and in the upper panel we show the rate of work done against the Lorentz force, \(-W_L\).
All values are normalized by $E_{M0}/T_s$, where $E_{M0}$ is the value of $E_M$ at $\tau=0$.

The rates of magnetic and kinetic energy dissipations, $\epsilon_M$ and $\epsilon_K$, respectively, can be read off as the difference between the two curves in each of the two panels in Fig. 10. Indeed, we have

$$-W_L - dE_M/dt = \epsilon_M,$$

(12)

$$W_L + W_C - dE_K/dt = \epsilon_K,$$

(13)

where the compressional work term $W_C=\int \rho \nabla \cdot \mathbf{U} \, dV$ is found to be negligible in all cases. Looking at Fig. 10 we can say that at early times ($0<\tau<0.7$) the magnetic field contributes to driving fluid motions ($W_L>0$) while at later times some of the magnetic energy is replenished by kinetic energy ($W_L<0$), but since magnetic energy dissipation still dominates, the magnetic energy is still decaying ($dE_M/dt<0$). The maximum dissipation occurs around the time $\tau=0.7$. The magnetic energy dissipation is then about twice as large as the kinetic energy dissipation. We note that the ratio between magnetic and kinetic energy dissipations should also depend on the value of the magnetic Prandtl number $P_{M}=\nu/\eta$, which we have chosen here to be unity. In this connection it may be interesting to recall that one finds similar ratios of $\epsilon_K$ and $\epsilon_M$ both for helical and nonhelical turbulence [26]. At smaller values of $P_{M}$ the ratio of $\epsilon_K$ to $\epsilon_K+\epsilon_M$ diminishes like $P_{M}^{-3/2}$ for helical turbulence [27]. In the present case the difference between $n=0$ and 2 is, again, small. Only at later times there is a small difference in $W_l$, as is shown in the inset of Fig. 10. It turns out that, for $n=2$, $W_l$ is positive while for $n=0$ its value fluctuates around zero. This suggests that the $n=2$ configuration is able to sustain fluid motions for longer times than the $n=0$ configuration. This is perhaps somewhat unexpected, because the helical configuration ($n=2$) should be more nearly force free than the nonhelical configuration. However, this apparent puzzle is simply explained by the fact that the $n=2$ configuration has not yet decayed as much as the $n=0$ configuration has.

IV. CONCLUSIONS

The present work has shown that the rate of magnetic energy dissipation is strongly constrained by the presence of magnetic helicity and not by the qualitative degree of knottedness. In our example of three interlocked flux rings we considered two flux chains, where the topology is the same except that the relative orientation of the magnetic field is reversed in one case. This means that the linking number switches from 2 to 0, just depending on the sign of the field in one of the rings. The resulting decay rates are dramatically different in the two cases, and the decay is strongly constrained in the case with finite magnetic helicity. The present investigations reinforce the importance of considering magnetic helicity in studies of reconnection. Reconnection is a subject that was originally considered in two-dimensional studies of X-point reconnection [28,29]. Three-dimensional reconnection was mainly considered in the last 20 years. An important aspect is the production of current sheets in the course of field line braiding [30]. Such current sheets are an important contributor to coronal heating [31]. The crucial role of magnetic helicity has also been recognized in several papers [32,33]. However, it remained unclear whether the decay of interlocked flux configurations with zero helicity might be affected by the degree of tangle-ness. Our present work suggests that a significant amount of dissipation should only be expected from tangled magnetic fields that have zero or small magnetic helicity, while tangled regions with finite magnetic helicity should survive longer and are expected to dissipate less efficiently.

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