A Heuristic Approach to a Non-Local Theory of Turbulent Channel Flow

In this paper a non-local relation is postulated to exist between the one-point-correlation tensor and the mean flow rate of strain. Thus the original ansatz of Boussinesq is replaced by an integral relation which naturally includes a wider region of the turbulent medium into the calculation. Using the idealization of free turbulence we can easily deduce that mean flow profile which has recently been given by Naue et al. and which appears to represent well the channel flow far away the walls. Within our integral formalism this profile proved to be correct for all those even kernels which provide exactly two solutions of a related characteristic equation. Testing some special kernels we find a good representation of the non-local formalism by a hydrodynamics involving in its stress relation the mean flow derivatives up to the third order.

The displacement between the Boussinesq and the hydrodynamical approach is a mere calculation of an additional integral term which expresses the difference between the exact and the approximate solution. The resulting integral equation is thus a non-local relation which includes the whole cross-section of the channel.

1. Introduction

REYNOLDS stress is defined as the negative product of density and the one-point-correlation tensor. Therefore, the structure of the latter tensor, which is symmetric by definition, plays the central part in any theory of the mean flow profiles observed in turbulent media. The usual way to state the correlation tensor — or its components required — consists in combining it with the mean flow. The coefficients of such relations are to be considered as semi-empirical quantities the best known of which is the so-called eddy viscosity defined by the special stress relation

\[ \overline{u'v'} = -r_T \frac{du}{dy} \quad \text{(Boussinesq)}, \]  

in case a parallel flow \( u \) shear over the \( y \)-coordinate. Some different expressions are suitable for \( r_T \):

\[ r_T = \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) \quad \text{(Prandtl)}, \]  

\[ r_T \approx \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \frac{\partial u}{\partial y} \right) \quad \text{(V. Karman),} \]  

representing non-linear stress relations.

The greatest defect of all of the formulae (1.1) consists in combining \( \overline{u'v'} \) only with the local gradient of the mean flow although we must expect the more distant regions to be of relevance, too [11]. One could try to introduce a slightly generalized expression instead of (1.1),

\[ \overline{u'v'} = -r_T^{(1)} \frac{du}{dy} + r_T^{(2)} \frac{d^2 u}{dy^2} \]  

which makes use of the next higher (odd) derivative. As we shall see later on, this relation permits a rather successful approach for interpreting the turbulent channel flow experiments. Indeed, the basic intuition of such an ansatz is given by its tendency to involve a larger vicinity into the consideration than the original Boussinesq theory.

2. On experiments

Recently Naue et al. [8] derived a representation of the mean flow profile within a turbulent channel by considering the turbulence as producing a Cauchy continuum. Consequently, the turbulence-originated part of the stress relation is assumed to consist of a tensor which is asymmetric with respect to its indices. Of course, the Reynolds theory of turbulence excludes such an approach because Reynolds stresses are related to a symmetric tensor:

\[ \frac{\partial u_i}{\partial t} + \langle u_j \rangle u_i = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + r u_i - \frac{\partial}{\partial x_j} Q_{ij}, \]  

where

\[ Q_{ij} = u_i(x, t) u_j(x, t) \]
is the one-point-correlation tensor. We cannot follow, therefore, the authors' basic outline but their analytic result seems to be rather useful. In the mentioned paper a two-dimensional channel is considered where the mean flow is along the $x$-direction and the walls are fixed at $y = \pm h$. The authors obtain

$$
u = \frac{\nu_m}{(y/h)^2 + \left(\frac{y}{h}\right)^2 (e^x + e^{-x} - 2) - (e^{y/h} + e^{-y/h} - 2)}$$

(2.3)

as the mean flow profile which fulfills the generally accepted conditions

$$\bar{u}(y) = 0, \quad \bar{u}(\pm h) = 0.$$  

(2.4)

The influence of the turbulence is concentrated in the last term of the RHS of eq. (2.3).

One can find this profile to be representing some of the known measurements of turbulent channel flows within a wide range of macroscopic Reynolds numbers. Næve et al. already claimed this fact and more recently Freigang [9] provided an extended numerical discussion. His results are listed in the following table (for $v = 0.155$ cm$^2$/s):

<table>
<thead>
<tr>
<th>$Re$</th>
<th>$h$ [cm]</th>
<th>$u_m$ [cm s$^{-1}$]</th>
<th>$\alpha$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reichenbach (1938)</td>
<td>7900</td>
<td>12.2</td>
<td>100</td>
<td>23.6</td>
</tr>
<tr>
<td>Laufer (1960)</td>
<td>12300</td>
<td>6.35</td>
<td>300</td>
<td>32.9</td>
</tr>
<tr>
<td>Laufer (1960)</td>
<td>30800</td>
<td>6.35</td>
<td>750</td>
<td>48.1</td>
</tr>
</tbody>
</table>

The constants $\alpha$ and $\gamma$ are always positive, and $\gamma > 1$. However, due to the relatively simple structure of (2.3) one cannot expect the profile to be exact in the walls where the turbulence must be considered inhomogeneous. Unfortunately, the authors failed to publish error limits calculations for the given parameters. In order to demonstrate the situation we show in Figs. 1 and 2 the experimental results by Eckelmann [2] in the usually used logarithmic representation which in particular underlines the region near the walls. The normalisations

$$u^+ = u/y^+,$$  

(2.5)

$$Y^+ = h^+(1 - y/h),$$

$$e^+ = \frac{1}{\nu} \frac{d}{dp}$$

(2.6)

Notice that the macroscopic Reynolds number is formed according to $R = \frac{u_m}{\nu} \cdot h^+$. Now we have at the wall $Y^+ = 0$ and $d\nu/dY^+ = 1$ so that by definition the laminar sublayer complies with $u^+ \approx Y^+$ (see [14]). The profile (2.3) takes then the form

$$u^+ = \frac{h^+}{\alpha(1 + 2\gamma)(e^x + e^{-x})} e^{\frac{2\gamma Y^+}{h^+} - \frac{\gamma Y^+}{h^+} + 1 - \exp\left(-\frac{\gamma Y^+}{h^+}\right)} + \frac{e^x}{e^{-x}} - 2\gamma Y^+ + \frac{\gamma Y^+}{h^+} + 1 - \exp\left(-\frac{\gamma Y^+}{h^+}\right),$$

(2.7)

![Fig. 1. Representation of the mean flow measurements by Eckelmann (600 cm$^{-1}$) and of the flow profile (2.7). Clearly, the actual value of $\alpha$ lies between 8 and 12. The normalized channel width $h^* = 142$](image1)

![Fig. 2. The same as in Fig. 1 for experiments with $h^* = 209$](image2)
which contains only one coefficient beside \( x \) and \( \gamma \). Near the wall we find

\[
\frac{u^+}{h^+} \approx (1 - \exp(-\alpha \gamma/h^+))/\alpha \tag{2.8}
\]

and at the channel center

\[
\frac{u}{x} \approx (1 + \gamma)/\alpha \tag{2.9}
\]

if really \( x \gg 1 \gg \gamma \). The coefficient \( \alpha \) determines thus the law of the wall whereas \( \gamma \) can be used for modelling the central region. Of course, the profile (2.7) does not imply the well-known and generally accepted logarithmic law of the wall \( u^+ \approx \ln Y^+ \) (see Figs. 1 and 2).

On the other hand, matching the profile to the observations concerning the central region \( Y^+ \leq h^+ \) does not present any problem. We shall accept here the Naive profile to give a correct representation of the observations in this region and demonstrate that a certain class of non-local hydrodynamics leads to exactly this solution without requiring any asymmetry of the stress tensor. To this end we have indeed to invoke only originally homogeneous turbulence fields so that deviations from the logarithmic law of the wall are not surprising. More generally speaking, the profile (2.3) proves to be one of the consequences of models with free turbulence. It might be questionable whether this idealisation can be used to produce an unnatural description of the wall situation.

3. The ansatz

In the following we shall discuss the consequences of postulating a generalized linear relation

\[
\overline{u'u'} = -\int g(y - y') \frac{du'(y')}{dy'} dy', \tag{3.1}
\]

where the kernel \( g \), which does not depend on any mean flow characteristics, shall possess the required convergency properties. It should particularly be emphasized that even the denotation \( g(y - y') \) implies some fundamental consequences concerning the physical situation. On the one hand the kernel is assumed to be a function only of the difference \( y - y' \) but not of the coordinates alone. By doing so we are restricting ourselves to homogeneous turbulence where the mean intensity is constant over the region considered. Thus, all theories based on symmetric kernels like our \( g \) cannot provide a very detailed analysis of the corresponding boundary layer problem.

Observe, on the other hand, that our kernel does not possess any temporal dependence. This seems to conflict with causality. We are discussing here, however, incompressible media whose velocity of sound is formally infinite so that all the retarding parts of the kernel vanish.

One is tempted to call a hydrodynamics which is based on such a relation a "non-local" one. Contrary to this the simple Boussinesq relation (1.1) where the kernel degenerates to Dirac's delta function must be considered as "quasi-local" whereas an ansatz like (1.2) could be classified as leading to "weakly non-local" hydrodynamics.

Obviously, the latter case can be obtained by taking a kernel

\[
g(y) = \nu^2 \delta(y) - \nu^2 \delta'(y), \tag{3.2}
\]

which provides a first step towards non-localness since it does not reflect a too trivial representation of the integral procedure (3.1).

Note that from this point of view the non-localness of a medium is given if the kernel \( g \) differs from Dirac's delta.

The proposed expression (3.1) must be understood as a specialisation of the theory of the influence of mean flow on turbulence or on its correlations (2.2). Thus, the existence of non-local relations seems to be quite natural in the same way as in the mean-field electro-dynamics [8], or stellar hydrodynamics [13], [15].

Assuming free turbulence, which may even exist for vanishing mean flow, Krause and Rudiger [5] have derived the general expression

\[
Q_{ij} = \frac{Q^{(0)}_{ij}}{\sigma} - \int \int \mathbf{N}_{ij}(x-x', \xi - \xi') \mathbf{\hat{u}}_{\xi, \xi}(x', \xi') \, dx' \, d\xi', \tag{3.3}
\]

where \( Q^{(0)}_{ij} \) represents the original correlation tensor for the given turbulence field uninfluenced by the mean flow and therefore not possessing nonvanishing cross components. In principle also nonlinear terms, i.e.\n\nonlinear terms in the mean flow \( \mathbf{u} \), may be included as was done in the papers by Rudiger [12] and Pope [9] — and we want to emphasize that there is indeed an ability of the turbulent media to become nonlinear in many physical situations. Here, however, we suggest the non-localness of the stress relation to be the more essential fact with regard to the channel flow theory.

4. Channel flow theory for non-specified kernels

From the considerations developed above we now want to formulate the channel flow theory on the basis of (3.1). The mean flow is defined to be along the \( x \)-coordinate and the plane-parallel walls of the channel are again fixed at \( y = \pm h \). No dependence on the coordinate \( z \) may occur. Thus we have

\[
\frac{d}{dy} u'u' = -\delta p + \frac{d^2 u}{dy^2}, \tag{4.1}
\]

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with uniform $\delta p$. Applying eq. (3.1) with infinite integration limits,

$$\overline{w'\varepsilon'} = -\int_{-\infty}^{\infty} g(y - y') \frac{du}{dy'} dy', \tag{4.2}$$

we get

$$-\int_{-\infty}^{\infty} g(y - y') U(y') dy' = -\delta p + \nu U(y), \tag{4.3}$$

where we introduced

$$U = \frac{d^2 \bar{u}}{dy^2} \tag{4.4}$$

as the basic quantity. By using infinite integration limits we have again ignored the real vanishing of the turbulence on the wall and outside the channel but not the vanishing of the cross correlation $w'\varepsilon'$ there, as we shall demonstrate later on. Hence, our analysis leads to an inhomogeneous Fredholm integral equation for $U$, the second derivative of the mean flow. Due to the infinite integration range this integral equation is a singular one for which reason no eigenvalue character originates. The inhomogeneity $\delta p$ of the equation induces a constant part $U_0$ for the second order derivative of the mean flow:

$$U_0 = \frac{\delta p}{\nu + \int g(y - y') dy'}. \tag{4.5}$$

As was to be expected the integral over $g$ plays the role of a global eddy viscosity.

In order to solve the integral equation completely we wish to find the situation for which the exponential form $U \sim \exp (\alpha y/\eta)$ represents a solution. Ignoring the inhomogeneity term leads to the remaining formula

$$\int_{-\infty}^{\infty} g(y - y') U(y') dy' + \nu U = 0. \tag{4.6}$$

This singular integral equation strongly reminds us of Picard's equation

$$g(z) = \lambda \int_{-\infty}^{\infty} e^{-i\eta z} g(z') dz', \tag{4.7}$$

which possesses the solutions $\exp (\pm i\mu x)$ with $\mu^2 = 1 - 2\lambda$ where $-1 < \Re \mu < 1$ (see [4]). Thus, real values $\mu$ exist for positive $\lambda$ only which would be equivalent to negative molecular viscosity $\nu$; hence kernels like Picard's one must be excluded. Bochner [1] showed that finite eigensolutions of (4.6) can generally be written as $\sum \exp (i\lambda_j \eta)$ and that the relation between the (real) $\lambda_j$ and $\nu$ is given by

$$\nu + \int_{-\infty}^{\infty} g(\eta) e^{-i\lambda \eta} d\eta = 0. \tag{4.8}$$

It is allowed, however, to drop the restriction on finiteness of the eigensolutions since we are merely interested in the relatively narrow channel region. Then the $\lambda_j$ become imaginary and we set $i\lambda_j = \kappa_j/h$ with real $\kappa_j$.

Let $y = \eta h$ and the condition (4.7) turns into

$$h\tilde{g}(\kappa_j) + \nu = 0, \tag{4.9}$$

where the tilde symbolizes the double Laplace transform taken over the region $[-\infty, \infty]$:

$$\tilde{g}(\kappa) = \int_{-\infty}^{\infty} g(\eta h) e^{-\kappa \eta} d\eta. \tag{4.9a}$$

From this formulation we at first conclude that the function $g$ must become negative in a sufficiently large region. In opposition to (4.9a) the Fourier transformation would be given by

$$\hat{g}(\kappa) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\eta h) e^{-i\kappa \eta} d\eta, \tag{4.9b}$$

and it holds $2\pi \hat{g}(0) = \tilde{g}(0)$. From the (say) $2\lambda$ roots of (4.8) we deduce as the general solution

$$U(\eta) = U_0 + \sum_{j=1}^{2\lambda} U_j e^{\kappa_j \eta}, \tag{4.10}$$

which corresponds to the mean flow

$$\bar{u}(\eta) = u_{-\lambda} + u_{-\mu} + u_{\mu} + \sum_{j=1}^{2\lambda} u_j e^{\kappa_j \eta}. \tag{4.11}$$

We shall always consider even kernels so that both the values $\kappa_j$ and $-\kappa_j$ solve the equation (4.8) and the number of independent values is reduced to $\lambda$. In this case the solution (4.11) can be easily applied to the channel
problem for which the conditions (2.4) must be fulfilled:
\[
\bar{u}(y) = u_m(1 - \eta^2) + \sum_{i=1}^{N} u_i(n_i^2(e^{\eta} + e^{-\eta} - 2) - (e^{n_\eta} + e^{-n_\eta} - 2)).
\]  (4.12)

(The free parameters \( u_i \) differ slightly from those in (4.11)). In this representation the influence of the turbulence is given by the sum again — it is represented by the number sets \( u_i \) and \( n_i \). The \( 1 - \eta^2 \) yields the well-known parabolic profile which characterizes laminar flow.

The expression (4.12) fulfills all the conditions concerning mean flow. It is necessary, however, to imply further conditions associated with the cross correlation \( \bar{u}'e' \) which must vanish at the walls. The crucial question is whether the constants \( u_i \) are able to produce the wanted behaviour at the boundaries. If this were so, a correct solution (2.3) could be provided without drastic restrictions on the homogeneous turbulence model. In particular the turbulence intensity need not vanish at the walls though the cross correlation does:
\[
\left. \bar{u}'e' \right|_{\eta=1} = \int g(h - \eta^2 h) \frac{dx}{dy} dy = 0. \]  (4.13)

This boundary condition can be fulfilled by fixing the parameters, indeed. To this end we insert the solution (4.12) into (4.13):
\[
\sum_{i=1}^{N} u_i \int (2\eta(e^{n_i} + e^{-n_i} - 2) - n_i(e^{n_\eta} - e^{-n_\eta})) g(h - h\eta) d\eta = 2u_m f \int g(h - \eta h) d\eta. \]  (4.14)

The calculation of the first integral is quite simple:
\[
\int_{-\infty}^{\infty} \eta g(h - \eta h) d\eta = 2\pi \tilde{g}(0) = \tilde{g}(0). \]  (4.15)

In order to determine the other integral one can use our basic condition (4.8):
\[
\int e^{\pm \eta^2} g(h - \eta h) d\eta = \int e^{\pm \eta^2(1 - \eta^2)} g(h\eta) d\eta = e^{\pm \eta^2} g(\pm \eta) = -\frac{d}{dh} e^{\pm \eta^2}. \]

Hence, the determining equation (4.14) of the wanted constants can be written as
\[
\sum_{i=1}^{N} u_i \left( 2\tilde{g}(0)(e^{n_i} + e^{-n_i} - 2) + \frac{\lambda_i}{h}(e^{n_i} - e^{-n_i}) \right) = 2u_m \tilde{g}(0). \]  (4.16)

Note the important role of the “eddy viscosity”
\[
\tilde{g}(0) = J g(h\eta) d\eta, \]  (4.17)

which implies that all turbulence models with vanishing \( \tilde{g}(0) \) are to be excluded. Otherwise the boundary condition (4.13) can really be fulfilled only by fixing the free parameters \( u_i \).

Concerning the number \( N \) we want to favour in this paper the simplest case, i.e. \( N = 1 \). Thus — after a short calculation — the profile of the mean flow takes the form
\[
\frac{\bar{u}}{u_m} = 1 - \eta^2 + 2\tilde{g}(0) \frac{(e^\eta + e^{-\eta} - 2)}{\lambda_1 e^{\eta}} + \frac{\lambda_1}{h}(e^\eta - e^{-\eta}) + 2\tilde{g}(0)(e^\eta + e^{-\eta} - 2). \]  (4.18)

This result strongly reminds us of the profile given by Naka et al. A comparison of the two expressions shows that eq. (2.3) is a solution for all even kernel functions \( g \) where \( N = 1 \) and
\[
\gamma = \frac{\eta}{2\tilde{g}(0)} > 0. \]  (4.19)

\( \gamma = \lambda_1 \) must satisfy the relation (4.8) as a real quantity. As the main result we can state that all non-local hydrodynamics with even \( g \) and \( N = 1 \) lead to the profile (2.3) describing the known observational facts correctly.

5. Two special kernels

In order to demonstrate the method of non-local stress relation in turbulent media a detailed example may be given now. A special kernel \( g \) must be postulated so that the exponents \( \lambda_i \) are explicitly deduced. The simplest \( g \)-function possessing the wanted properties seems to be:
\[
g = \left( \psi_1^2 - 2 \frac{\psi_2^2}{\psi_1^2} \right) \text{\( \delta \)}(y) \equiv \left( \psi_1^2 - 2 \frac{\psi_2^2}{h^2 \psi_1^2} \right) \text{\( \delta \)}(\eta). \]  (5.1)

Note the (symbolic) negative values of this function whose Laplace transform is simple and even, indeed:
\[
g = \frac{1}{h} \left( \psi_1^2 - \frac{\psi_2^2}{h^2 \psi_1^2} \right). \]  (5.2)
Following eq. (4.8) one obtains the values

\[ \chi = \alpha , \quad \alpha = \hbar \frac{\sqrt{u + v^2}}{v_p^2} . \]  

\[ (N = 1) , \]  

and the quantity \( \gamma \) is

\[ \gamma = \frac{v}{2v_p^2} . \]  

Therefore, the task to reconstruct the profile (4.18) due to Nau \( \text{e} \) is very simple if the above concept of non-local hydrodynamics is used. Both the turbulent parameters, \( v_T^{(1)} \) and \( v_T^{(2)} \), are uniform throughout the whole channel region.

It is quite interesting that the special kernel (5.1) allows a corresponding differential formulation, namely

\[ v_T^{(2)} U'' = - \delta p + (v + v_T^{(1)}) U , \]  

or, which is the same,

\[ v_T^{(2)} \frac{d^2 u}{dy^2} = (v + v_T^{(1)}) \frac{d p}{d y} . \]  

This is really the equation with which Nau \( \text{e} \) et al. have operated in their paper — being based, however, on a completely different concept of turbulence. In the Appendix we shall demonstrate the singular part of the quantity \( v_T^{(2)} \) in constructing a general one-point-correlation tensor including higher derivatives of the mean motion. Channel flow experiments are suitable, as we have seen, to measure this exclusive quantity.

The integral problem which is based on the special kernel (5.1) relates to a differential problem of an order enhanced by two in comparison to the laminar case. Now we will suggest a kernel which would produce a differential problem of infinite order justifying the non-localness in a more general sense. Let us consider the more realistic form

\[ g(y) = \frac{1}{l^{2\pi}} \left( \frac{v_T^{(1)}}{l} + 2 \frac{v_T^{(2)}}{l^2} - 4 \frac{v_T^{(3)} y^2}{h^2} \right) e^{-v_T^{(2)} y} , \]  

which is identical with our first example (5.1) in the case of \( l = 0. \) Laplace transformation provides

\[ \tilde{g}(\alpha) = \frac{1}{h} \left( \frac{v_T^{(1)}}{v_T^{(2)}} - \frac{v_T^{(3)} x^2}{h^2} \right) e^{\frac{\alpha x}{h^2}} . \]  

Applying (4.8) we derive the relation

\[ v + \left( \frac{v_T^{(1)}}{v_T^{(2)}} - \frac{x^2}{h^2} \right) e^{\frac{\alpha x}{h^2}} = 0 . \]  

Again we have \( N = 1 \) with

\[ \chi = \alpha , \]  

where

\[ \chi = \hbar \frac{\sqrt{u + v^2}}{v_p^2} \quad \text{if} \quad l^2 \gg \frac{v_T^{(2)}}{v_T^{(1)}}, \quad \text{or} \quad \chi = \hbar \frac{v + v_T^{(1)}}{v_T^{(2)}} \quad \text{if} \quad l^2 \ll \frac{v_T^{(3)}}{v_T^{(1)}} . \]  

As was expected, sufficiently small lengths \( l \) lead to the same \( \chi \) as that for the simplest kernel (5.1). Moreover, even for large lengths the differences between these two cases are quite small for \( v_T^{(1)} \gg v \) (small \( \gamma \)). Consequently the result does not strongly depend on the length scale \( l \). This is the reason why we cannot find any information about \( l \) from the analysis of the experiments. Their full information concerns the quantities \( v_T^{(1)} \) and \( v_T^{(2)} \) according to the relations

\[ v_T^{(1)} = \frac{v}{2 \gamma}, \quad v_T^{(2)} = \frac{v h^2}{2 \gamma^2} . \]  

From the measurements quoted above we find the respective values

\[
\begin{array}{cc}
\text{Reichardt (1938)} & 4.87 & 0.887 \\
\text{Lauffer (1950)} & 5.07 & 0.199 \\
\text{Lauffer (1950)} & 15.2 & 0.265 \\
\end{array}
\]

Finally we can state from these results that the higher order hydrodynamics (1.2) — based on a non-local stress relation — may be considered as a good approach to treat turbulent channel flows. The simplifications used, i.e. in particular the implied nonvanishing of the turbulence outside the channel, presumably can be overcome by a more realistic specification of the integrating procedure.
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Appendix

Let us construct the one-point-correlation tensor in its linear dependence on the mean flow. The general expression is

\[ Q_{ij} = Q_{ij}^{(0)} + N_{i j} \bar{u}_k - N_{i j} \bar{u}_k, t + N_{i j k} \bar{u}_{ij} t + N_{i j k l m} \bar{u}_{ij}, t m + \ldots \]

with

\[ Q_{ij}^{(0)} = Q_{ii}^{(0)}, \quad N_{ij} = N_{ji}, \]

where all the quantities \( Q_{ij}^{(0)} \) as well as \( N_{ij} \) should be considered as independent of \( \bar{u} \). They represent the general physical situation, therefore including all the geometrical quantities necessary for the characterisation of this situation. For example, such geometrical quantities could be given by the gravitation vector, angular velocity or even the \( y \)-direction if the two walls of a channel were made of different materials. In case the turbulent field does not possess any special features besides the mean flow itself, only KRONNECKER tensors \( \delta_{ij} \) are available for the construction. Thus only tensors with an even number of indices can occur. Already having considered the symmetry with respect to \( i \) and \( j \) we find in all generality

\[ Q_{ij}^{(0)} = a \delta_{ij}, \]

\[ N_{ij} = \epsilon_{ij} \delta_{kl} \bar{u}_k \delta_{ij} + \mu \hat{\delta}_{ij} \delta_{ij}. \]

For incompressible media, \( \bar{u}_k = 0 \), the \( \delta_{ij} \)-term is not of relevance for the forming of \( Q_{ij} \). Furthermore,

\[ N_{ijk} = a_1 (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl}) \delta_{mn} + (a_2 \delta_{ij} \delta_{jk} + \delta_{ij} \delta_{km}) \delta_{ln} + (\alpha_1 \delta_{ij} \delta_{km} + \delta_{kl} \delta_{ij} \delta_{mn}) \delta_{ln} + \]

\[ + (\alpha_2 \delta_{ij} \delta_{km} + \delta_{kl} \delta_{ij} \delta_{ml} + \delta_{jk} \delta_{im} \delta_{mn}), \]

Here terms with \( \delta_{kl} \) as well as those with \( \delta_{km} \) and \( \delta_{kn} \) do not contribute to the correlation tensor for incompressible media. Now we can introduce

\[ a_1 = \epsilon_{ij}^{\nu} - 2, \quad a_2 = \epsilon_{ij}^{\nu} + \mu_1 + \mu_2, \quad \alpha_1 = \epsilon_{ij}^{\nu} - \mu_2, \]

which yields

\[ N_{ijk} = \epsilon_{ij}^{\nu} \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} \right) \delta_{mn} + \left( \delta_{ij} \delta_{jk} + \delta_{ij} \delta_{km} \right) \delta_{ln} + \left( \delta_{ij} \delta_{km} + \delta_{ij} \delta_{kl} \delta_{mn} \right) \delta_{ln} + \]

\[ + \mu_1 \left( \delta_{ij} \delta_{jm} \delta_{km} \delta_{ln} - \delta_{ij} \delta_{kl} \delta_{lm} \delta_{mn} \right) + \]

\[ + \mu_2 \left( \delta_{ij} \delta_{jm} \delta_{km} \delta_{ln} - \delta_{ij} \delta_{kl} \delta_{lm} \delta_{mn} \right) \cdots \delta_{kl} + \ldots \delta_{km} + \ldots \delta_{kn}. \]

Due to their asymmetry also the tensors with \( \mu_1 \) and \( \mu_2 \) do not contribute to the correlations. Therefore, all of the tensorial possibilities only the \( \epsilon_{ij}^{\nu} \)-term (besides \( \epsilon_{ij}^{\nu} \)) can emerge within the correlation tensor or within the REYNOLDS equation, respectively.

References


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