Big Bang Nucleosynthesis Limits and Relic Gravitational Waves Detection Prospects

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We revisit the big bang nucleosynthesis (BBN) limits on primordial magnetic fields and/or turbulent motions accounting for the decaying nature of turbulent sources between the time of generation and BBN. This leads to larger estimates for the gravitational wave (GW) signal than previously expected. We address the detection prospects through space-based interferometers (for GWs generated around the electroweak energy scale) as well as pulsar timing arrays and astrometric missions (for GWs generated around the quantum chromodynamics energy scale).

Gravitational radiation from the early universe propagates almost freely throughout the universe’s expansion and primordial gravitational waves (GWs) reflect a precise picture of the universe at their time of generation, ranging from a tiny fraction of the first second to the first three minutes after the Big Bang. Detection of these GWs is a promising tool that would open new avenues to understand physical processes at energy scales inaccessible to high energy particle physics experiments but accessible to astrophysical observations; see Ref. [1] for a review.

There are several milestones of modern cosmology, proven through CMB anisotropies and large scale structure statistics; including the hot beginning of the universe, flatness of space-time to high precision, preservation of isotropy (rotation symmetry) and homogeneity of the universe at scales almost comparable with today’s Hubble horizon, see Ref. [2] and references therein. However, crucial physical processes in the very early universe remain unknown. In particular, the light element abundances allow us to reconstruct the picture of big bang nucleosynthesis (BBN) but leave the matter-antimatter asymmetry (baryogenesis) question open [3, 4]. Unknowns in model building prior to BBN include the low number of e-folds during inflation, hypothetical particles (including sterile neutrinos, axion-like particles, and dark radiation) that might serve as solutions for the dark matter puzzle, etc; see Ref. [5] for a review. These unknowns will be reflected in the variety of relic GW characteristics, including not only the strength of the signal and its spectral shape, but also its polarization. Indeed detection of GW polarization is a unique option to test fundamental symmetries at these extremely high energies. If the GWs originated from parity violating sources in the early universe, they will be circularly polarized and, unlike the CMB, GW polarization will exist at the basic background and not just the perturbation level; see Ref. [6] for pioneering work and see Refs. [7–12] for recent studies. This phenomenon is analogous to the GWs produced via Chern-Simons coupling [13, 14]. If detected, the GW polarization can be a direct measure of the deviations from the standard model (SM) [15–17]. One of the major goals of this Letter is to determine whether these circularly polarized GWs (and their polarization) are potentially detectable in the upcoming early-universe GW observation missions [18].

BBN data (based on light element abundances) impose an upper limit on the universe’s expansion rate, e.g. the Hubble parameter, \( H(t_{\text{phys}}) = (d\ln a/dt_{\text{phys}}) \) (with physical time \( t_{\text{phys}} \) and scale factor \( a(t_{\text{phys}}) \)), and correspondingly, on the additional relativistic species such as massless (or ultrarelativistic) hypothetical particles, early stage dark energy (or any bosonic massless field), dark radiation, electromagnetic fields or early-universe plasma motions (turbulence), relic GWs, etc [19–22]. Conventionally, the energy density of these additional relativistic components is characterized in terms of the effective number of relativistic species, \( N_{\text{eff}} \). The SM predicts an effective number of neutrino species \( N_{\text{eff}}^{(\nu)} = 3.046 \), which is slightly larger than 3 because neutrinos did not decouple instantaneously and were still able to interact with photons and electrons near electron-positron annihilation [20]. Other additional relativistic components contribute \( \Delta N_{\text{eff}} = N_{\text{eff}} - N_{\text{eff}}^{(\nu)} \) to this effective neutrino count. Notably, the presence of additional relativistic components does not spoil the time dependence of the scale factor during the radiation-dominated epoch, but it does affect the Hubble parameter and Hubble time scale, \( H^{-1} \). The joint analysis of CMB measurements and BBN light element abundances put \( N_{\text{eff}} = 2.862 \pm 0.306 \) at 95% confidence [21]. Using the upper bound of this error interval (\( N_{\text{eff}} = 3.168 \)), we express the maximum ra-
tio of additional components of energy density $\rho_{\text{add}}$ to the radiation energy density $\rho_{\text{rad}}$ at the BBN temperature as $\rho_{\text{add}} / \rho_{\text{rad}} \approx 0.0277 \cdot (\Delta N_{\text{eff}}/0.122)$, normalized around $\Delta N_{\text{eff}} = 0.122$ (corresponding to this $N_{\text{eff}}$ value).

The maximum value of this ratio is limited by the combined CMB and BBN data. We note that this upper bound coincides with the constraint on the GW contribution to the radiation energy density found by Ref. [28] using CMB and BBN data combined with limits from NANOGrav and late-time measurements of the expansion history. Interestingly, the light element abundances (with the bounds on $N_{\text{eff}}$) impose limits on the lepton asymmetry in the universe [29] that might result in primordial chiral magnetic fields [30] and correspondingly serve as a source for polarized GWs [12].

In this Letter we address the BBN bounds from the point of view of early-universe anisotropic stress (namely primordial magnetic fields and turbulent sources) and the induced GW signal. We are particularly interested in the strength, the spectral shape, and the polarization degree of the induced GWs. Violent processes in the early universe might lead to the development of turbulence. In particular, first order electroweak and quantum-chromodynamic (QCD) phase-transition bubble collisions and nucleation might lead to turbulent plasma motions [31–34], or, alternatively, turbulence can be induced by primordial magnetic fields coupled to the cosmological plasma [35–38]. The stochastic GW background from these turbulent sources has been studied for decades now; see Refs. [39] for pioneering works and Ref. [40] for a review and references therein. Recently the GW signal and its polarization have been studied in Refs. [41–44] (for space based interferometers) or through the Earth surface curvature (for ground-based interferometers) recently explored in Ref. [45] and mostly referring to GWs generated at and around the electroweak energy scale [51]. Despite promising detection prospects for stochastic GWs through pulsar timing arrays (PTAs), which are potentially sensitive to GWs generated around the QCD energy scale, detection of the polarization degree looks to be problematic. Detection of circular polarization has been ruled out for an isotropic GW background, but may be possible in the case of an anisotropic background [41–44]. However, the signal requires many pulsar observations ($\geq 100$, a number achievable by the International Pulsar Timing Array [52]) and very large signal-to-noise ratios ($\geq 400$) [53].

Due to weak coupling between gravity and matter (i.e., the smallness of Newton’s constant $G$), the GW generation from any turbulent source is characterized by low efficiency ($\mathcal{E}_{\text{GW}}/\mathcal{E}_{\text{turb}} \ll 1$, where $\mathcal{E}_{\text{GW}}$ is the GW energy density) and, consequently, the ratio between turbulent source energy density and total radiation energy ($\zeta$) is not affected by emission of gravitational radiation. In other words, the energy radiated in GWs will not induce substantial damping of the turbulent energy density. Moreover, if turbulent decay processes are discarded (i.e., velocity and magnetic fields are “frozen-in” to the primordial plasma), $\zeta$ is unchanged during the radiation-dominated epoch. Applying this logic to the BBN bounds, the few percent limit was applied a priori to much earlier time-scales when GWs were generated. As it was seen in simulations [11–44], the GW energy density reaches a maximum and stays unchanged after a short time. Thus, only $\zeta$ at the moment of the source activation (i.e., GW generation) matters.

In the case of decaying turbulence, the situation is different: the $\zeta$ parameter is time dependent and the decay rate is determined by the specific model of turbulence (helical vs. non-helical, magnetically or kinetically dominant, etc). Decaying turbulence leads to a power-law decay of the (magnetic or kinetic) energy density, $\mathcal{E}_{\text{turb}}(t) \propto (t/t_*)^{-p}$, and growth of the correlation length $\xi_{\text{turb}}$ of the field by an inverse cascade mechanism such that $\xi_{\text{turb}} \propto (t/t_*)^q$, where $t = \int dt\text{phys}/a$ is the con-

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1. There are several motivations to consider early-universe turbulent sources and primordial magnetic fields, see Ref. [39] for a review and references therein. Additionally there are several manifestations of parity symmetry violations in astrophysical objects, including one-sided, oriented radio jets that might be an indication of helical (chiral) magnetic fields present in the early universe [34].
FIG. 1: Possible turbulent evolution of the comoving magnetic field strength $B$ and correlation length $\eta_M$ from generation at the electroweak phase transition (EWPT) and QCD scale in the cases of fully helical ($\beta = 0$), nonhelical ($\beta = 1, 2, 4$), and partially helical (with $\epsilon_{M, *} = 10^{-3}$) MHD turbulence. Upper limits on the correlation length are determined by the size of the horizon and number of domains (bubbles) at generation, ranging from 1 to 6 (at QCD) or 100 (at EWPT). Lines terminate (on the right) at recombination ($T = 0.25$ eV). The upper limit of the comoving field strength at BBN ($T = 0.1$ MeV) is indicated by the black dot-dashed line. Regimes excluded by observations of blazar spectra [54] are marked in gray. Each hatched region is bounded by an upper line corresponding to the (upper) limit from BBN and a lower line corresponding to the (lower) limit from the blazar spectra.

formal time and the parameters $p$ and $q$ depend on the properties of the turbulence (e.g., in helical turbulence $p = q = 2/3$, while for non-helical magnetically dominated turbulence $p = 1$ and $q = 1/2$, but other variants are possible). The scaling exponent $q$ may reflect the presence of an underlying conservation law (helicity conservation, Loitiansky integral) and is also determined by the nature of turbulence (kinetically or magnetically dominated). The combined values of $p$ and $q$ for a particular process can be summarized by the parameter $\beta$ such that $p = (1 + \beta)q$ [55], where $\beta$ characterizes the decay of the spectral peak of magnetic energy. Partially helical magnetic fields are also described by their fractional helicity, i.e., the ratio of the magnetic helicity to its maximal value, $\epsilon_{M, *} < 1$. Due to this decay, the BBN bound allows larger values of $\zeta$ at the moment of generation, making the GW signal stronger. The limits at the moment of GW generation depend on the decay process duration – the time elapsed from the moment of generation until the BBN epoch. Thus, the maximum allowed energy density of turbulent sources that satisfy the BBN limits will be different at the electroweak and QCD energy scales (e.g., electroweak turbulence has a longer decay period, allowing higher values for the initial energy density that still satisfies the BBN bounds). Figure 1 shows the bounds on the strength of the magnetic fields at their generation (electroweak or QCD scales) determined such that the strength does not exceed the upper limit of the comoving field strength at BBN (see [60]) and is above the lower observational bounds on the field strength at recombination (at a temperature of 0.25 eV).

As we can see from Fig. 1, allowed values for the magnetic fields at the moment of generation are not constrained to microGauss field strength, as it was claimed previously based on BBN bounds without accounting for decaying turbulence [40]. In fact, if previously we were considering the Alfvén speed (or plasma motion characteristic velocity) around 0.2–0.3 (in units of the speed of light), the new limits possibly imply $v_A(v_T) \rightarrow 1$ [44]. Obviously, in this case we deal with relativistic turbulence that might be characterized by different decay laws or efficiency to generate GWs. However, recent relativistic turbulence numerical simulations [57] show that the basic properties of turbulence decay are preserved, including non-helical inverse cascading. Also, following arguments of Ref. [41], the non-relativistic description of turbulent sources results in an underestimation of the signal.

Below we present the first simulations of the GW signal from such strong turbulence sources. We use the PENCIL Code [58, 59] to simulate MHD turbulence in the early Universe by computing the stochastic GW background and relic magnetic fields [44]. In all cases, turbulence is driven by applying an electromagnetic force that is $\delta$-correlated in time and has the desired spatial spectrum. We vary the forcing strength and adjust the viscosity such that the smallest length scales in the simulation are sufficiently well resolved to dissipate the injected energy near the highest available wavenumber. We perform runs...
for the QCD and EW energy scales; see Ref. [60] for a table summarizing the eight runs presented in this paper.

The GW detection prospects are strongly affected by the characteristic frequency ranges and thus the energy-containing wave number of the source. More precisely, the GW spectrum peaks at the comoving angular frequency \( \omega_{\text{peak}} = (2\pi f_{\text{peak}}) = 2k_1 \), where \( k_1 \) is the initial peak wave number of the source energy density spectrum (in natural units \( c = 1 \)). The inertial wave number is determined by the turbulent eddy size \( (k_1 = 2\pi/L) \), and if we assume that turbulence arises from phase transitions, the eddy size may be associated with the bubble size \( 60 \). Independently of the nature the turbulence, the typical length scale is limited by the Hubble scale. In what follows, we use the characteristic wave number \( k_0 \) normalized by the Hubble wave number \( H_* \).

The energy density of early-universe turbulent sources is determined by the efficiency of converting the available radiation energy into kinetic or magnetic energies. In the case of first-order phase transitions, it can be expressed in the terms of the parameter \( \alpha = \rho_{\text{vac}}/\rho_{\text{rad}} = 4\rho_{\text{vac}}/3(\rho + P) \) (with \( \rho \) and \( P \) being the plasma energy density and pressure, respectively) – the ratio between the latent heat (false vacuum energy density) and the plasma radiation energy density (which is determined at the phase transition temperature \( T^\alpha \)). \( \alpha \sim a \) few corresponds to extremely strong phase transitions. Ref. [61] discusses a few beyond-SM models, which could include first-order phase transitions, and some of these models predict \( \alpha \gtrsim 1 \) for specific ranges of their parameter spaces. In particular, the addition of a 6-dimensional term to the Higgs potential \( 62 \) or the addition of a singlet scalar field \( 63 \) allow for these particularly strong phase transitions. The induced turbulence can then be characterized by the velocity field, \( v_i = 1/\sqrt{(1 + (\rho + P)/(2E_i)} \), where \( v_i \) refers either to kinetic motion velocity \( v_T \) (associated with fluid motions) or the effective Alfvén velocity \( v_A \) (associated with magnetic fields) and \( E_i \) refers to either the kinetic, \( E_K \), or magnetic, \( E_M \), energy density, and \( k\equiv \kappa(\alpha) \in (0, 1) \) is the efficiency coefficient (that increases with \( \alpha \)), i.e., the fraction of vacuum energy that is transformed into fluid kinetic (\( E_K \)) or magnetic (\( E_M \)) energy, rather than into heat \( 61 \). This formulation allows us to recover relativistic expressions for turbulent motions, \( v_T = 1/\sqrt{1 + 4/(3\kappa)} \) \( 64 \), and the Alfvén velocity, \( v_A = 1/\sqrt{1 + (4/3)//(2E_M)} \) \( 62 \), \( 2 \) while previous studies (see Ref. [43] for a review and references therein) assumed non-relativistic motions.

The additional relativistic degrees of freedom in the early universe due to the addition of the energy densities of the turbulent sources can be subsumed into \( \Delta N_{\text{eff}} \). This increase in \( N_{\text{eff}} \) increases the CMB-inferred value of the Hubble constant, \( H_0 \), helping to reduce the tension with late-universe values. A value of \( \Delta N_{\text{eff}} \sim 0.4 \) could alleviate the Hubble tension \( 60 \). Interestingly, it has been shown that the recent NANOGrav results may also favor a larger value of \( N_{\text{eff}} \) \( 67 \) if the signal arises in the early universe. Even though the large values of \( \alpha \) parameter (\( \sim \) few) are not restricted by currently available BBN or other observational data we limit ourselves by \( \alpha \leq 1 \) that was addressed previously in several studies, see Ref. [1] and references therein.

In Fig. 2 we present GW spectra from our simulations expressed as \( h_{\text{eff}}^2\Omega_{\text{GW}}(f_{\text{phys}}) \) for two families of models: one for the electroweak phase transition with \( k/H_* = 600 \) and one for the QCD phase transition with \( k/H_* = 6 \). The former set of models is similar to simulations of Ref. [6], except that now we also consider models with stronger turbulent driving which is applied over one Hubble time along with a period during which the forcing decreases linearly in time to zero, again over one Hubble time.

As already noted in previous studies \( 11, 12, 44 \), the GW energy spectrum from forced turbulence shows almost no or a rapidly declining inertial range for frequencies above the peak. This is because only the smallest wave numbers contribute significantly to the driving of GWs \( 44 68 \). In fact, the GW energy \( h_{\text{eff}}^2\Omega_{\text{GW}}(f_{\text{phys}}) \) scales approximately quadratically with the ratio of magnetic energy to characteristic wave number \( k_0 \) as \( (qE_M/k_0)^2 \), where \( q \) is the efficiency (of order unity). For the QCD phase transition, the characteristic wave number is a hundred times smaller, so the GW energy is correspondingly larger.

Toward smaller frequencies, the spectra show a shallower fall-off, in some cases proportional to \( f_{\text{phys}}^2 \). This is steeper than what has been found in earlier simulations at lower magnetic energies, but shallower than what was generally expected based on analytical considerations. Physics beyond the SM often leads to parity symmetry breaking and correspondingly to polarized gravitational waves. In Fig. 3 we show the polarization spectra for the same runs as in Fig. 2. For the QCD phase transition with only a few bubbles per linear Hubble scale, the polarization spectra have an extended region with \( P_{\text{GW}} \sim 1 \), while for the electroweak phase transition with tens of bubbles, the polarization spectra have non-trivial profiles with a narrower plateau.

In summary, the BBN data does not limit the total energy density of turbulence (kinetic or magnetic energy density) at the moment of its generation to be 10% of the radiation energy when the decay process is accounted for. Strong turbulence unavoidably results in a more powerful source for the GW signal with more optimistic prospects for GW detection.

Data availability—The source code used for the simulations of this study, the PENCIL CODE, is freely available from Refs. [58, 59]. The simulation setups and the corresponding data are freely available from Ref. [60].

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2 In the non-relativistic velocity limit we obtain usual expressions \( v_T = \sqrt{2E_K/(\rho + P)} \) and \( v_A = B_{\text{eff}}/\sqrt{4\pi(\rho + P)} = \sqrt{2E_M/(\rho + P)} \).
FIG. 2: Frequency spectra, $h_2^2\Omega_{GW}(f)$, for both the QCDPT Runs a–d (left) and the EWPT Runs A–D (right) shown in red, orange, blue, and black, respectively.

FIG. 3: Polarization spectra, $P_{GW}(f)$, for the QCDPT Runs a–d (left) and the EWPT Runs A–D (right) shown in red, orange, blue, and black, respectively.

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Supplementary Material to “Big Bang Nucleosynthesis Limits and Relic Gravitational Waves Detection Prospects”

I. NUMERICAL SET-UP/GRAVITATIONAL WAVES

We consider the radiation-dominated epoch at electroweak (EW) and quantum chromodynamic (QCD) energy scales and compute the strains \(h_+\) and \(h_\times\) for the two linear polarization modes by solving the linearized equation for gravitational waves (GWs),

\[
\frac{\partial^2}{\partial t^2} h_+/\times + k^2 h_+/\times = \frac{6}{a} \tilde{T}_+/\times,
\]

where \(\tilde{T}_+/\times\) are the + and \(\times\) polarizations of the Fourier transform of the total stress \(T_{ij} = u_i u_j - B_i B_j\) normalized by the radiation energy density, with \(t\) and \(k\) the time and wave vector normalized by the Hubble parameter at the time of generation, and \(B = \nabla \times A\) and \(u\) are obtained by solving the equation for the magnetic vector potential

\[
\frac{\partial A}{\partial t} = u \times B + \eta \nabla^2 A,
\]

together with \(33\)

\[
\frac{\partial u}{\partial t} = -u \cdot \nabla u - \frac{1}{4} \nabla \ln \rho + \frac{3}{4\rho} J \times B + F_\nu + F,
\]

\[
\frac{\partial \ln \rho}{\partial t} = -\frac{4}{3} (\nabla \cdot u + u \cdot \nabla \ln \rho) + H,
\]

where \(F = (\nabla \cdot u + u \cdot \nabla \ln \rho) u - \frac{1}{4} u (J \times B) + J^2 / \sigma |u| / \rho\), and \(H = |u| (J \times B) + J^2 / \sigma \rho\) are higher order terms in the Lorentz factor that are retained in the calculation, and \(F_\nu = 2 \nabla \cdot (\rho \nu S) / \rho\) is the viscous force, where \(S_{ij} = \frac{1}{2} (u_i u_j + u_j u_i) - \frac{1}{3} \delta_{ij} \nabla \cdot u\) are the components of the rate-of-strain tensor with commas denoting partial derivatives, and \(\nu\) is the kinematic viscosity. In all cases considered below, we assume a magnetic Prandtl number of unity, i.e., \(\nu / \eta = 1\). In Table I we summarize the parameters for runs a–d and A–D for the QCD and EW energy scales, respectively.

As in Ref. \[11\], hereafter K+21, we compute GW generation from magnetically driven turbulence. The driving is applied during the time interval \(1 \leq t \leq 2\), where \(t\) is the conformal time. As in K+21, we then decrease the driving linearly in time until \(t = 3\), when the driving is turned off completely. We perform series of runs where we vary the strength of the forcing \(f_0\) and keep the viscosity \(\nu\) unchanged. However, it is not possible to explore the regime of strong magnetic energy at the same small values of \(\nu\) that we were able to use for smaller magnetic energies. This is because for strong magnetic fields, the turbulence becomes more intense and more viscosity is needed to dissipate all this energy at the finite numerical resolution available.

In Fig. 4 we show the resulting dependence of the GW energy \(\epsilon_{GW}\) on the magnetic energy \(\epsilon_M\) for six sets of runs with fixed viscosity, different forcing strengths, and different forcing wavenumbers, corresponding to the runs denoted with labels a–d, A–D, and O. In all cases, we take the magnetic Prandtl number to be unity, i.e., the magnetic diffusivity is set equal to the value of \(\nu\). We also compare with several other sets of runs where we change the forcing.

In Table I we summarize the parameters for four runs (A–D), which correspond to the less viscous ones for each of the four pairs shown in Fig. 4. One exception is Run D, which has the same viscosity as Run C and is denoted in
Fig. 6: Evolution of (a) $\nu$ and (b) $\nu$ for Runs A–D of Table I. Note the rapid decay for Run A with the largest viscosity.

The data for $\nu$ follow a power law scaling, $\nu \propto \nu^n$, where $n = 2.7$ for the points with the smallest viscosity. This is steeper than the quadratic scaling found in the work of [14], where the driving was applied for a much shorter time interval, $1 \leq t \leq 1.1$. Furthermore, for fixed values of $\nu$, we find smaller local values of $\nu$, at least for the larger magnetic energies shown in Fig. 6. We also checked that these scalings are not significantly affected if the driving was turned off abruptly after $t = 2$. This is shown as the dotted line in Fig. 6 for $\nu = 5 \times 10^{-5}$.

Comparing the lines for $\nu = 5 \times 10^{-5}$ and $\nu = 10^{-4}$ in Figs. 4 and 5, we see that the decline of $\nu$ is stronger than that of $\nu$. This suggests that $\nu$ suffers more strongly from the increase of viscosity and magnetic diffusivity, and that $\nu$ is less sensitive to the change of $\nu$. However, one has to remember that GWs are solely the result of the magnetic and hydrodynamic stresses. One sees that the runs with smaller values of $\nu$ all have a faster rise of $\nu$ early on, which also translates into a rapid increase of $\nu$. It is unclear, however, whether this aspect of the model with applied magnetic driving is realistic and whether this would also be borne out by a more physical implementation of a magnetogenesis model.

Next, we show in Fig. 6 the evolution of $\nu$ and $\nu$ with time. We see that for Runs C and D, $\nu$
has reached a plateau well before $t = 2$, while for Run A, a maximum is reached only at $t = 2$, i.e., the time when the driving is decreased. Moreover, for Run A, there is a strong temporal decline of magnetic energy due to strong viscous damping. Nevertheless, similar GW energies are obtained in this case. The value of $E_{GW} = 3 \times 10^{-5}$ given in Table I corresponds to $h_0^2 \Omega_{GW} = 4.93 \times 10^{-10}$, which is four orders of magnitude larger than for Run D.

II. MAGNETIC FIELD BOUNDS

The bound on extra relativistic degrees of freedom at big bang nucleosynthesis (BBN) can be expressed as

$$\frac{\rho_B(T_{BBN})}{\rho_\gamma(T_{BBN})} = f, \quad (5)$$

where we have assumed that all the extra relativistic energy density is entirely due to the magnetic energy density $\rho_B$. $\rho_\gamma$ is the energy density in photons, $T_{BBN}$ is the temperature at the onset of BBN, and $f \equiv \frac{4}{\pi} (\frac{4}{3})^{1/3} \Delta N_{\text{eff}}$.

The photon energy density as a function of temperature is $\rho_\gamma = (\pi^2/15) T_4^4$. The magnetic energy density is related to the magnetic field strength $B$ as $\rho_B = B^2/8\pi$ (in Gaussian units). The magnetic field strength dilutes with the expansion of the universe as $B \sim a^{-2}$ where $a$ is the cosmological scale factor. The comoving magnetic field strength is given by $B^c_\star = (a/a_0)^2 B(a)$, where $a_0$ is the scale factor today. Substituting these values into the equation, the BBN limit on the field strength today is given by

$$B^c_\star \leq \left( \frac{a_{BBN}}{a_0} \right)^2 \sqrt{8 \pi f \rho_\gamma(T_{BBN})}. \quad (6)$$

Obtaining the ratio of the scale factors via entropy conservation, normalizing such that $a_0 = 1$, the bound is given by

$$\frac{B^c_\star}{\text{Gauss}} \leq (8.06 \times 10^{-6}) f^{1/2} g_{BBN}^{-2/3} \quad (7)$$

where $g_{BBN}$ is the relativistic degrees of freedom at $T_{BBN}$. There is no explicit dependence on temperature, however, the total number of relativistic degrees of freedom $g_{BBN}$ does depend on the temperature. At $T_{BBN} = 0.1$ MeV, the temperature at which deuterium synthesis starts, neutrinos, electrons, and positrons have already decoupled and $g_{BBN}(T = 0.1\text{MeV}) \approx 3.4$. For $\Delta N_{\text{eff}} = 0.122$, we find $f = 0.028$ and the maximum comoving field strength at BBN is $B^c_{BBN} = 6.2 \times 10^{-7}$ G.